Symposium On Geometry Processing 2010

Möbius Transformations For Global Intrinsic Symmetry Analysis

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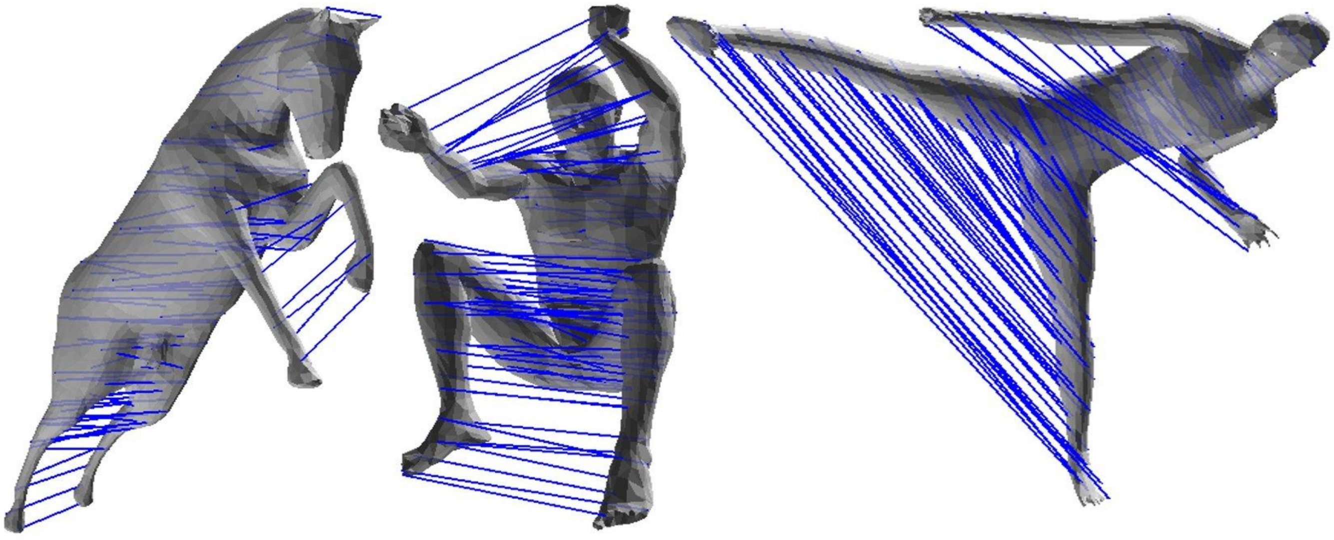
Princeton University

Goal



- Find a map f from surface onto itself that preserves geodesic distances

$$f : \mathcal{M} \rightarrow \mathcal{M} \ \ \text{s.t.} \ \ d_g(p,q) = d_g(f(p), f(q))$$

Previous Work

- **Extrinsic Symmetry**
- Intrinsic Symmetry
  - Symmetry Axis
  - Laplace-Beltrami Operator
  - Gromov-Hausdorff Distance
- Inter-Surface Correspondence
  - Möbius Voting

Podolak et al., 2006

Mitra et al., 2006

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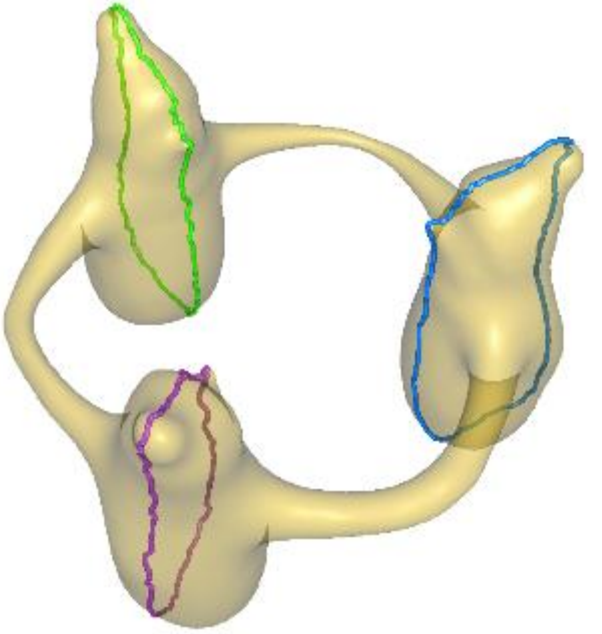

Xu et al., 2009

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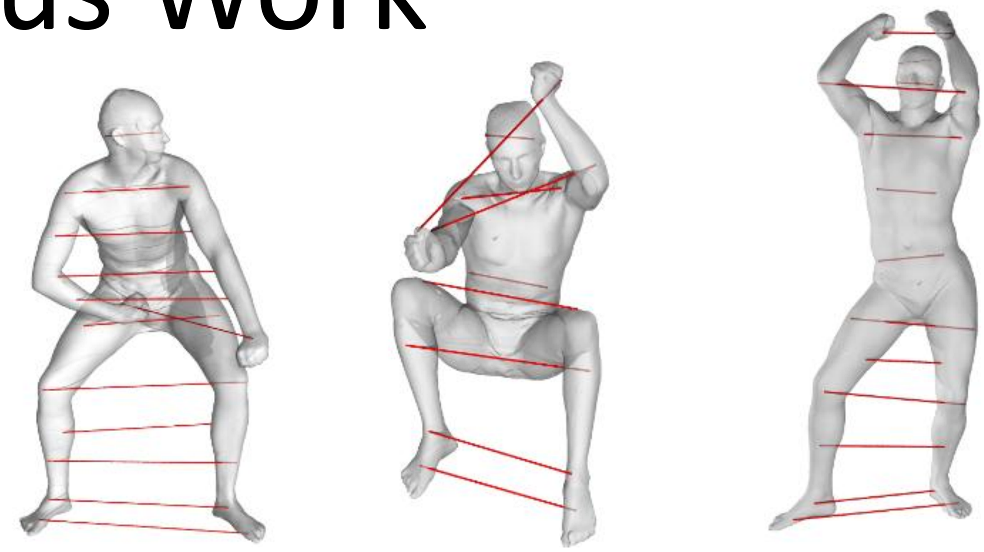

Ovsjanikov et al. '08

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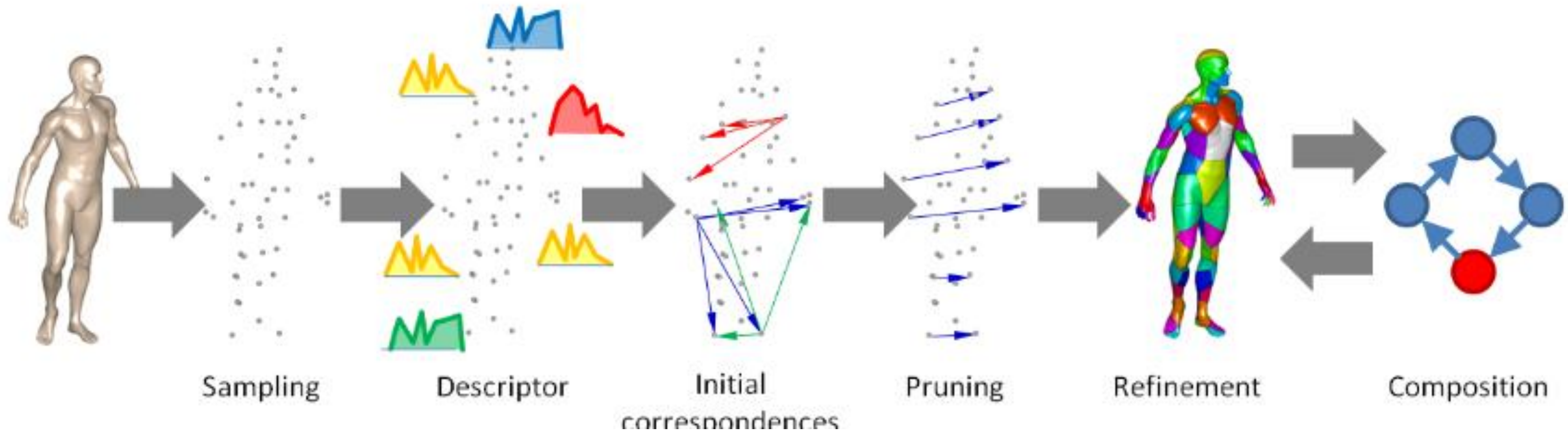

Raviv et al. '10

- **Gromov-Hausdorff Distance**

- Inter-Surface Correspondence
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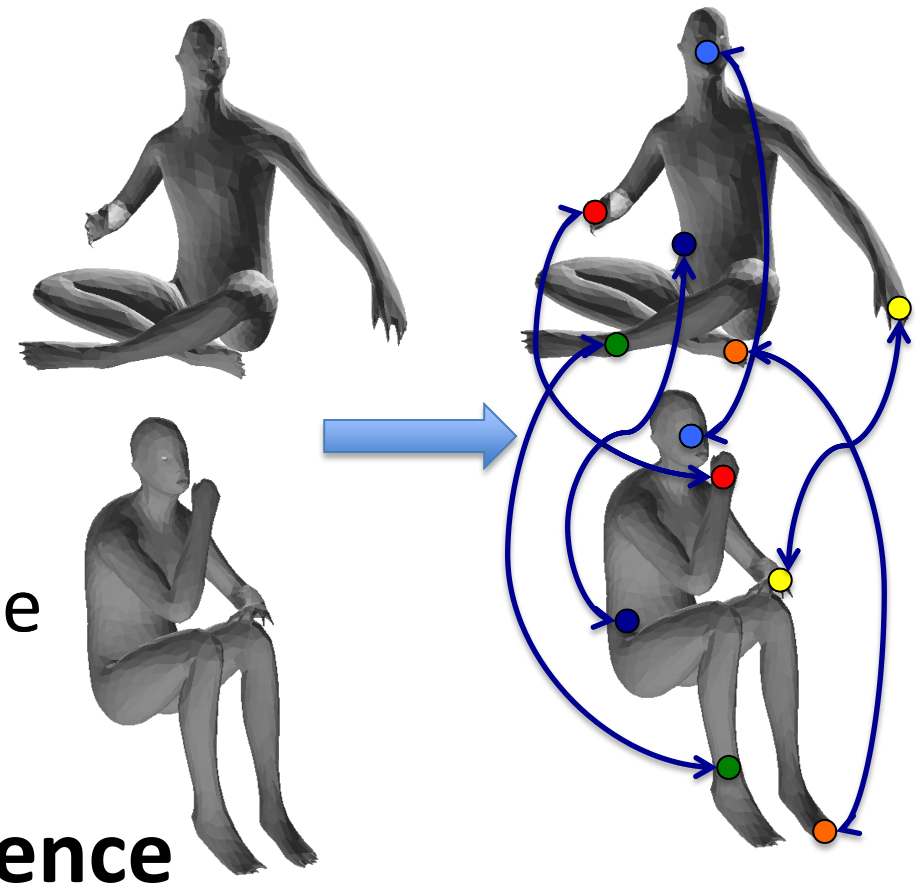
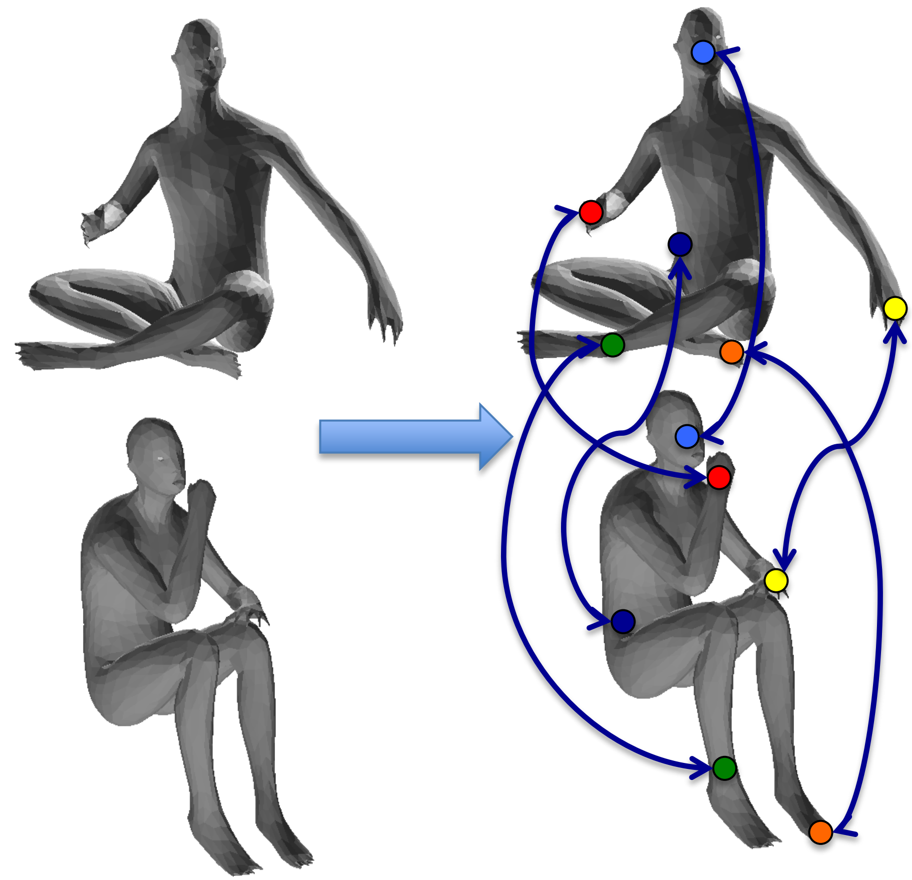


- **Inter-Surface Correspondence**
  - **Möbius Voting**

Previous Work
Möbius Voting

- **Look for an isometry**
  - Conformal
  - Area-preserving
- Conformal Maps
  - Mid-edge flattening
  - Möbius Transformation
  - Defined by 3 correspondences



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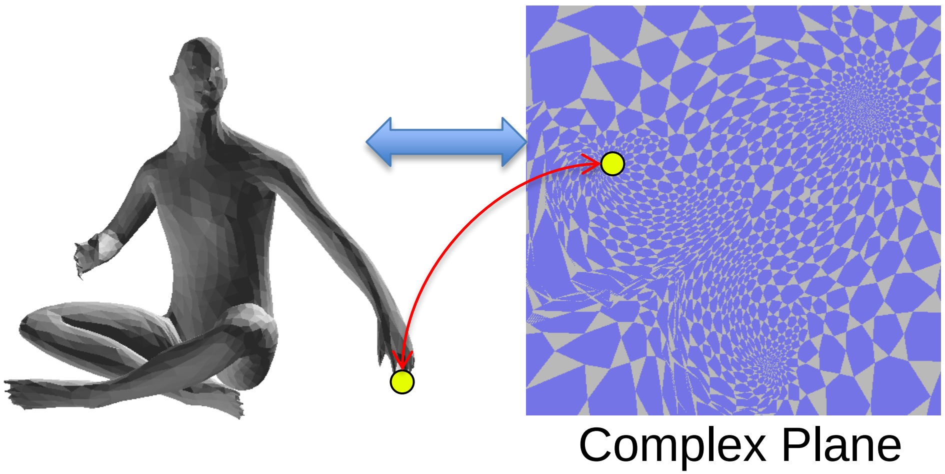

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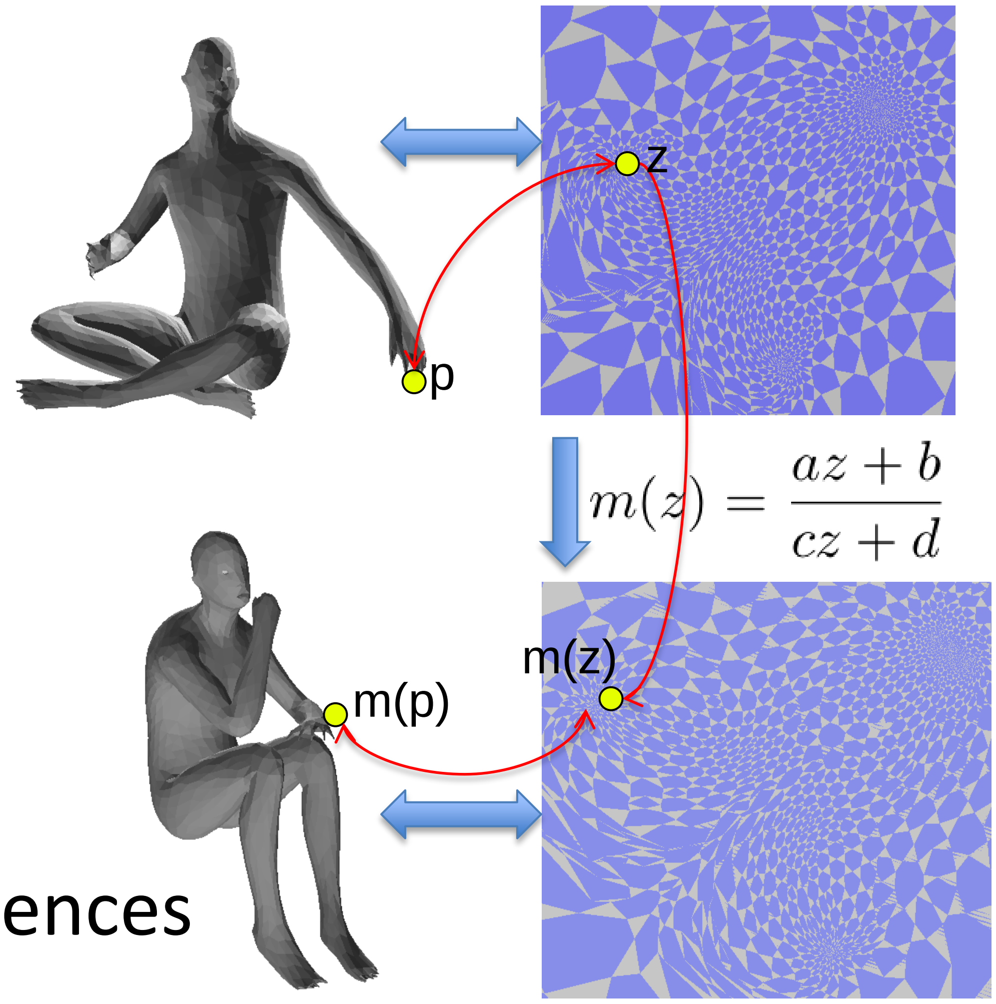

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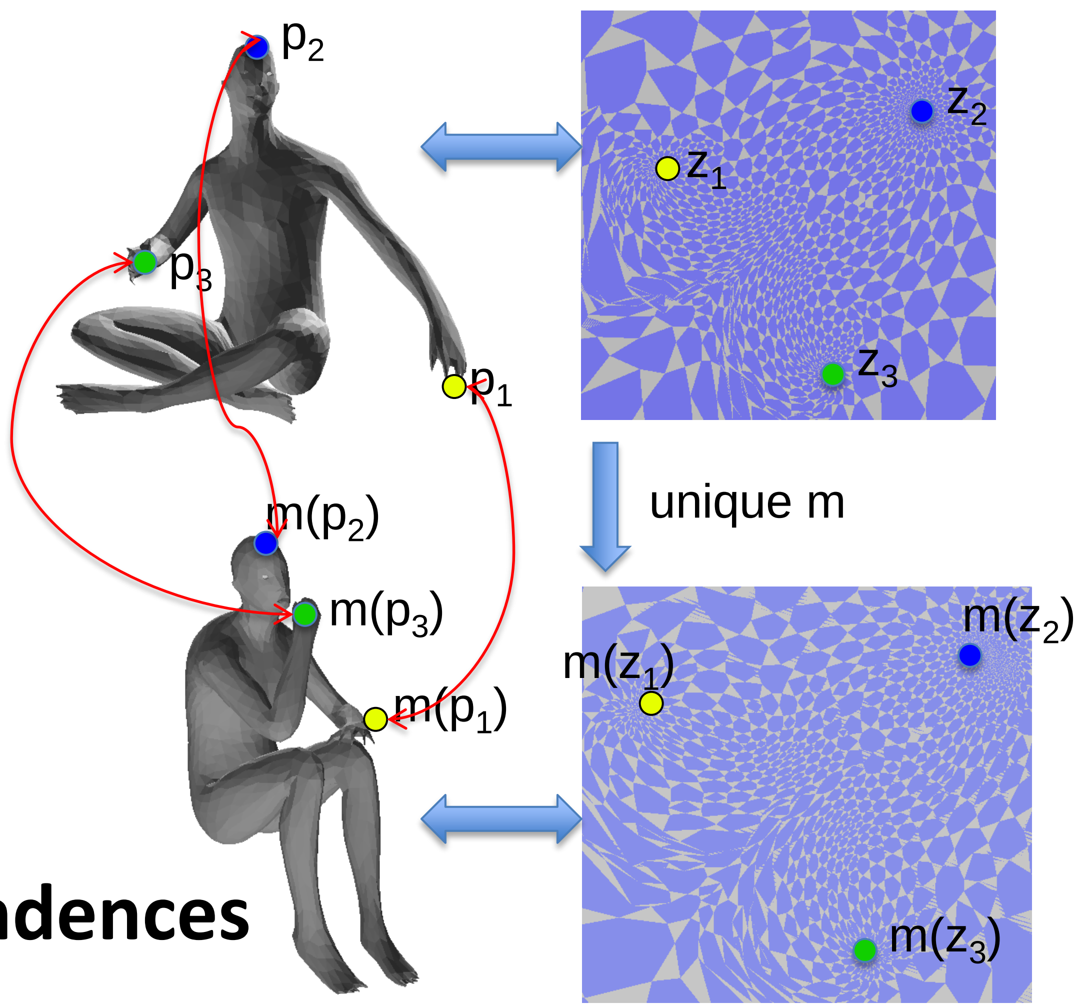
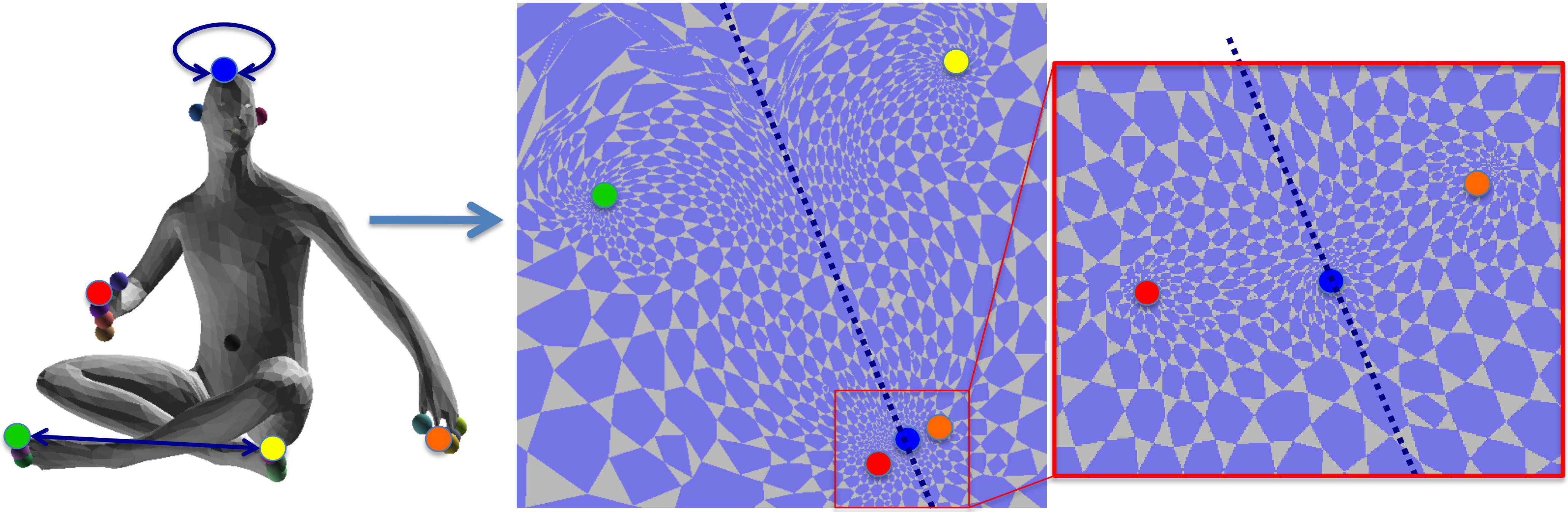
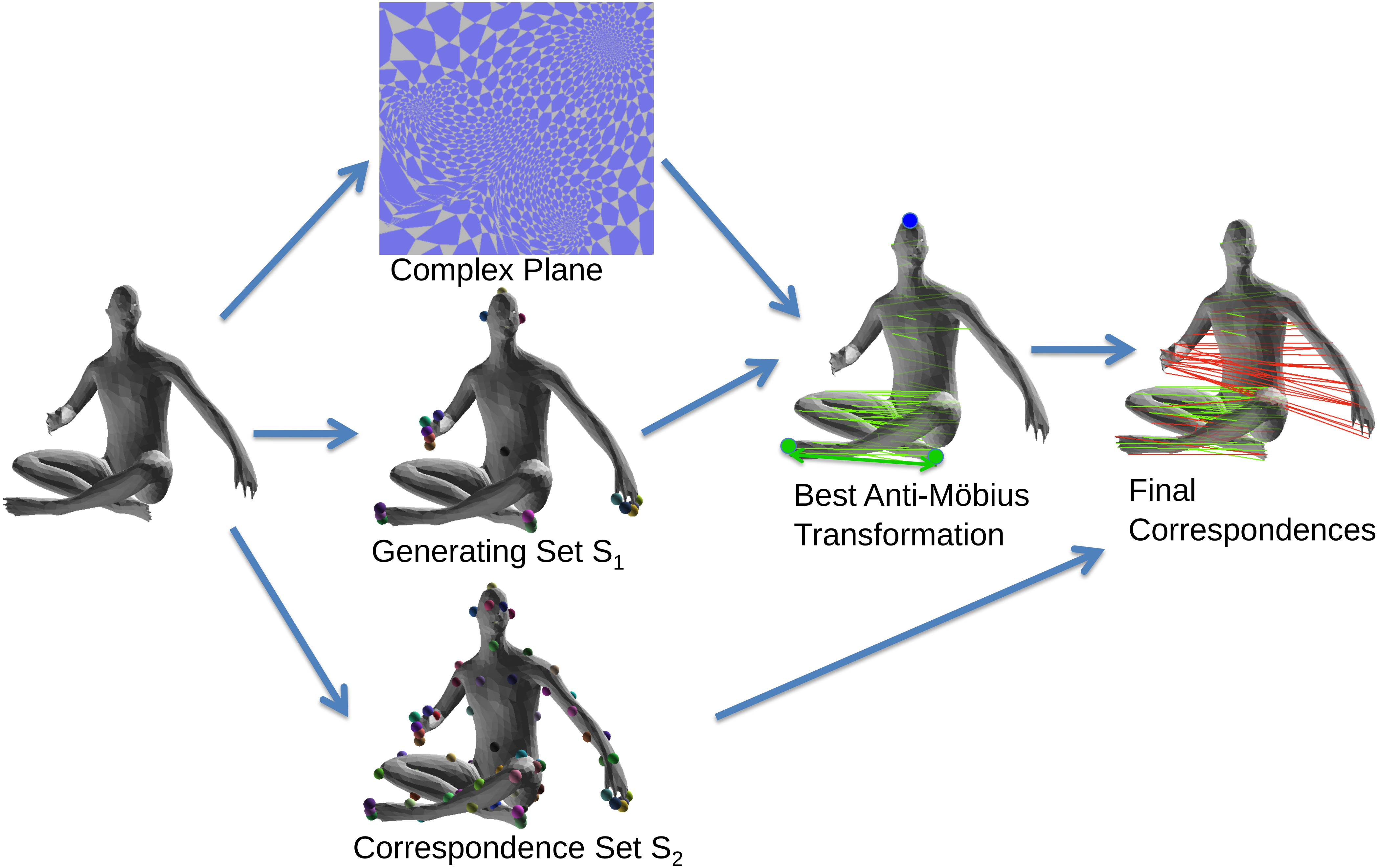


Our Approach

- Look for an Anti-Möbius Transformation that makes intrinsic symmetry extrinsic on complex plane



Pipeline


Complex Plane
Generating Set S_1
Correspondence Set S_2
Best Anti-Möbius
Transformation
Final
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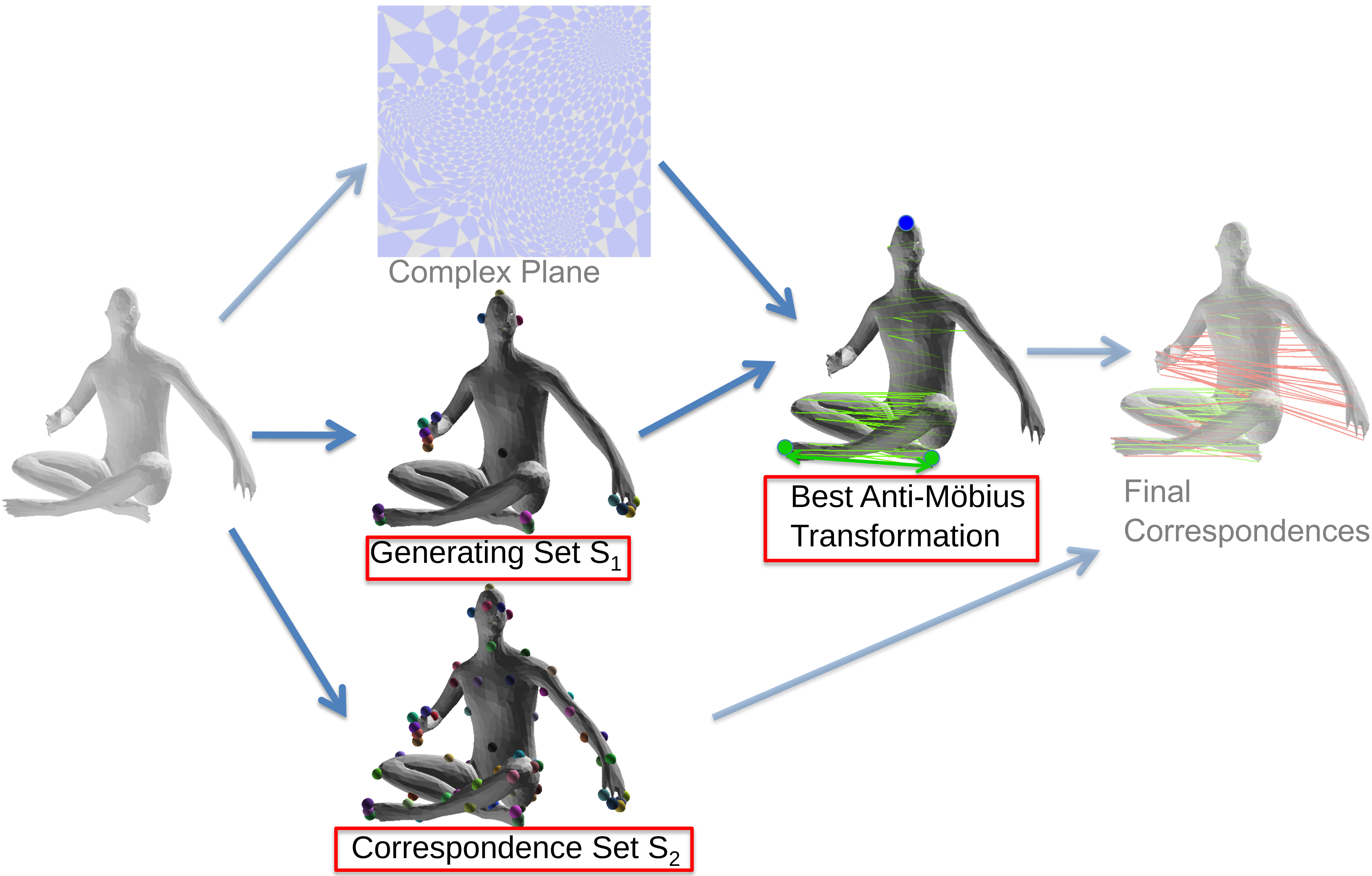

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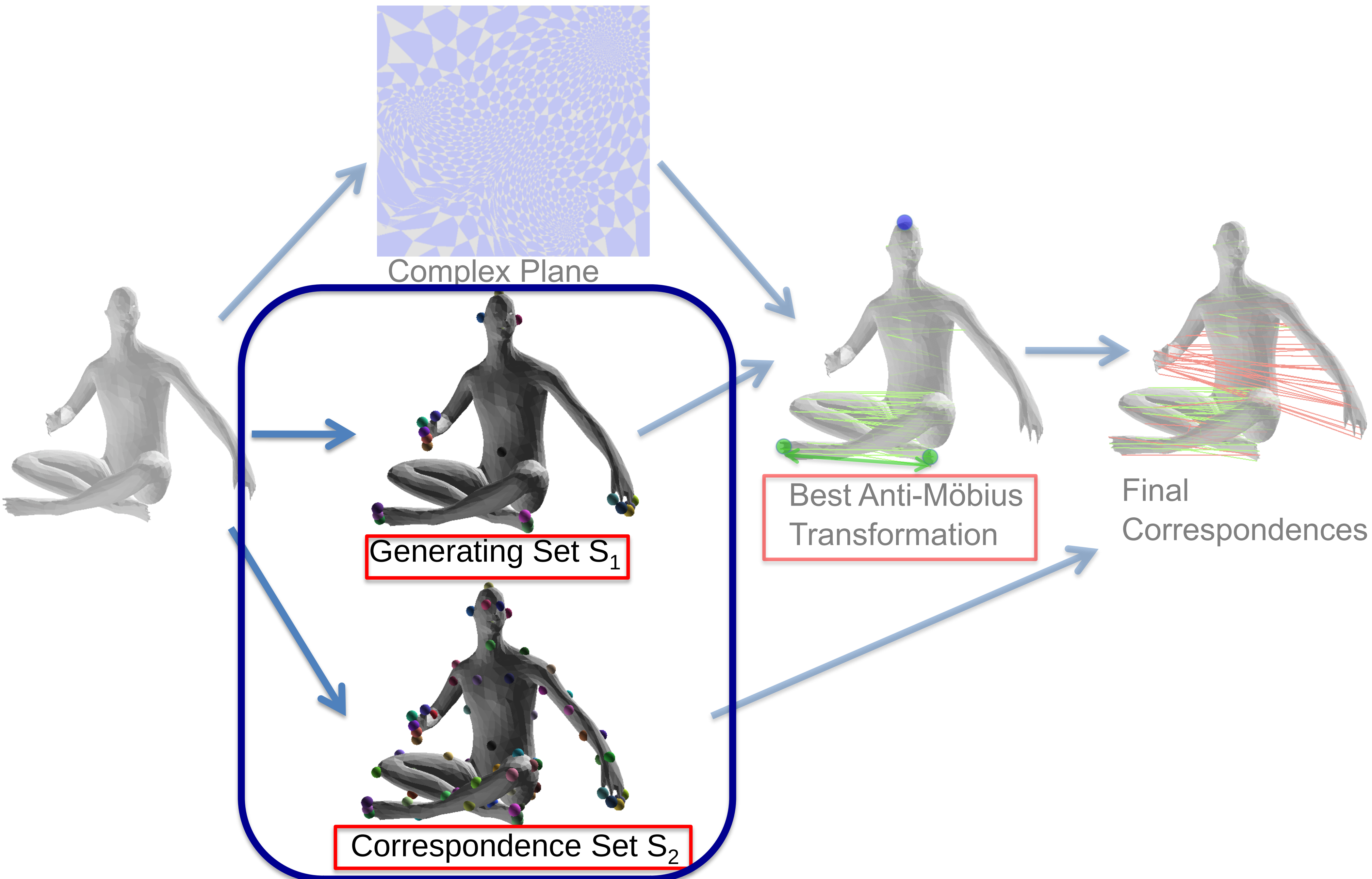
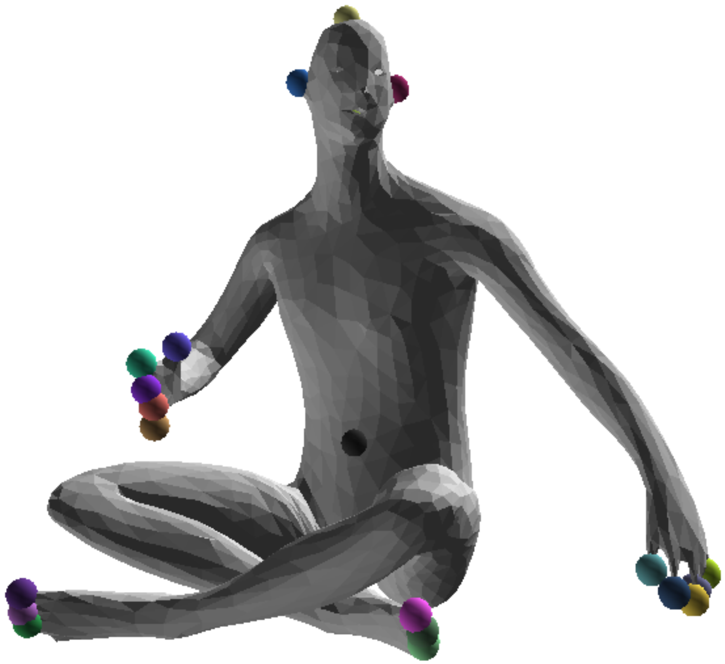
Finding a Symmetric Point Set



Finding a Symmetric Point Set

- Goal: need a set containing potential correspondences and stationary points

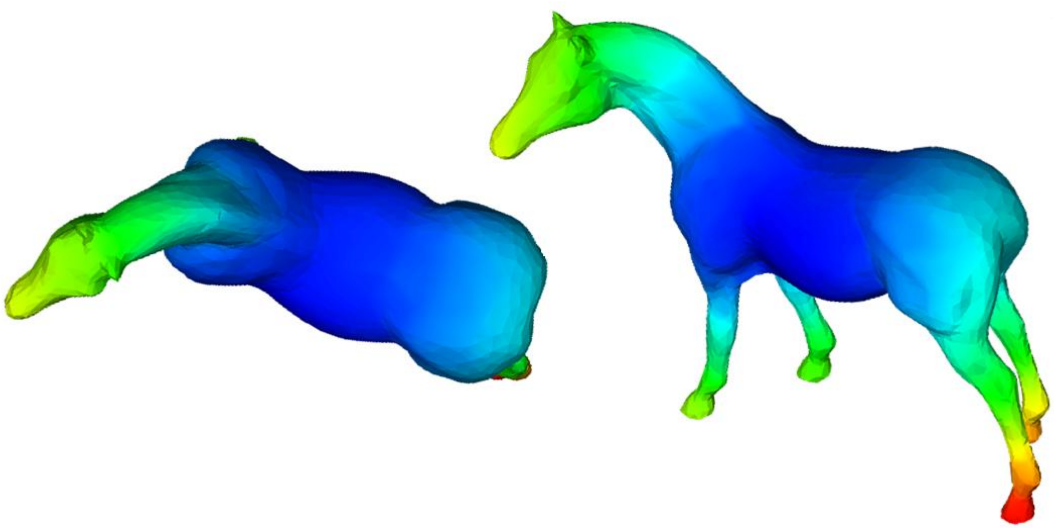
  e.g. Find a set $S \subset \mathcal{M}$ invariant under f : $f(S) = S$



- Approach: use critical points of symmetry invariant function Φ

Finding a Symmetric Point Set
Example Symmetry Invariant Function

- Average Geodesic Distance $\Phi_{\mathrm{agd}}(p) = \int_{d_g(p,q)} d_g(p,q)dq$



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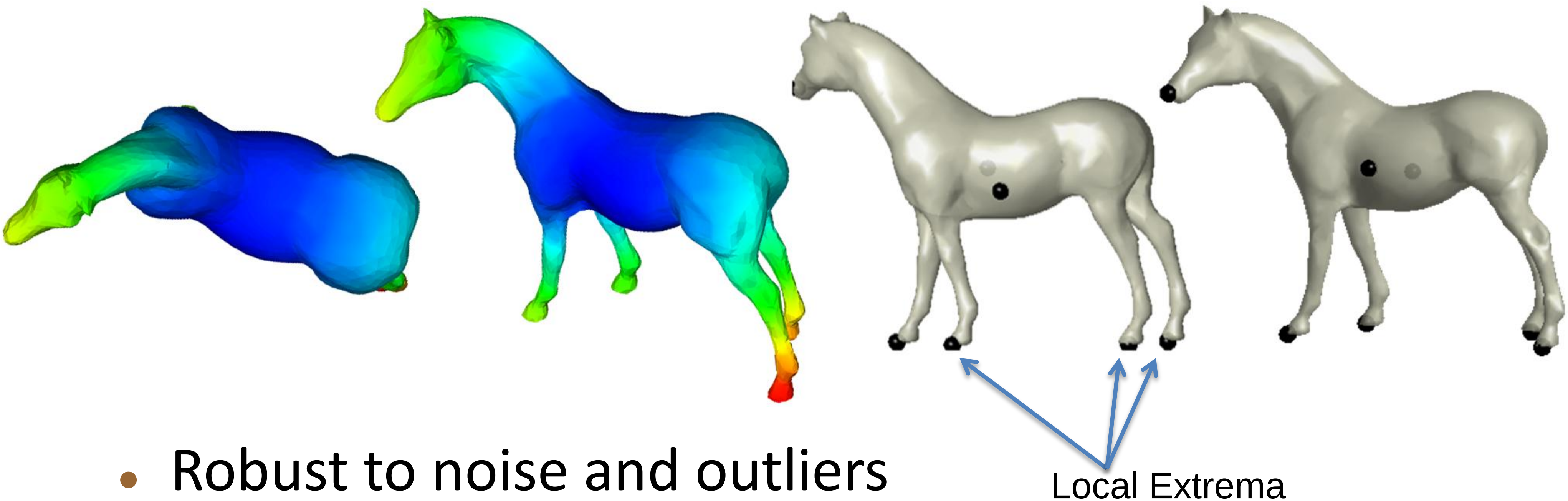

- Robust to noise and outliers
- Only few extrema
- Generating Set for Anti-Möbius Transformations

Finding a Symmetric Point Set Theory

- Symmetry: $f : \mathcal{M} \rightarrow \mathcal{M} \text{ s.t. } d_g(p, q) = d_g(f(p), f(q))$

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  Look for critical points

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  - If f is bilateral reflective, the gradient of Φ is parallel to the curve of stationary points of f

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  At least 2 stationary points will have $\nabla|_p\Phi=0$

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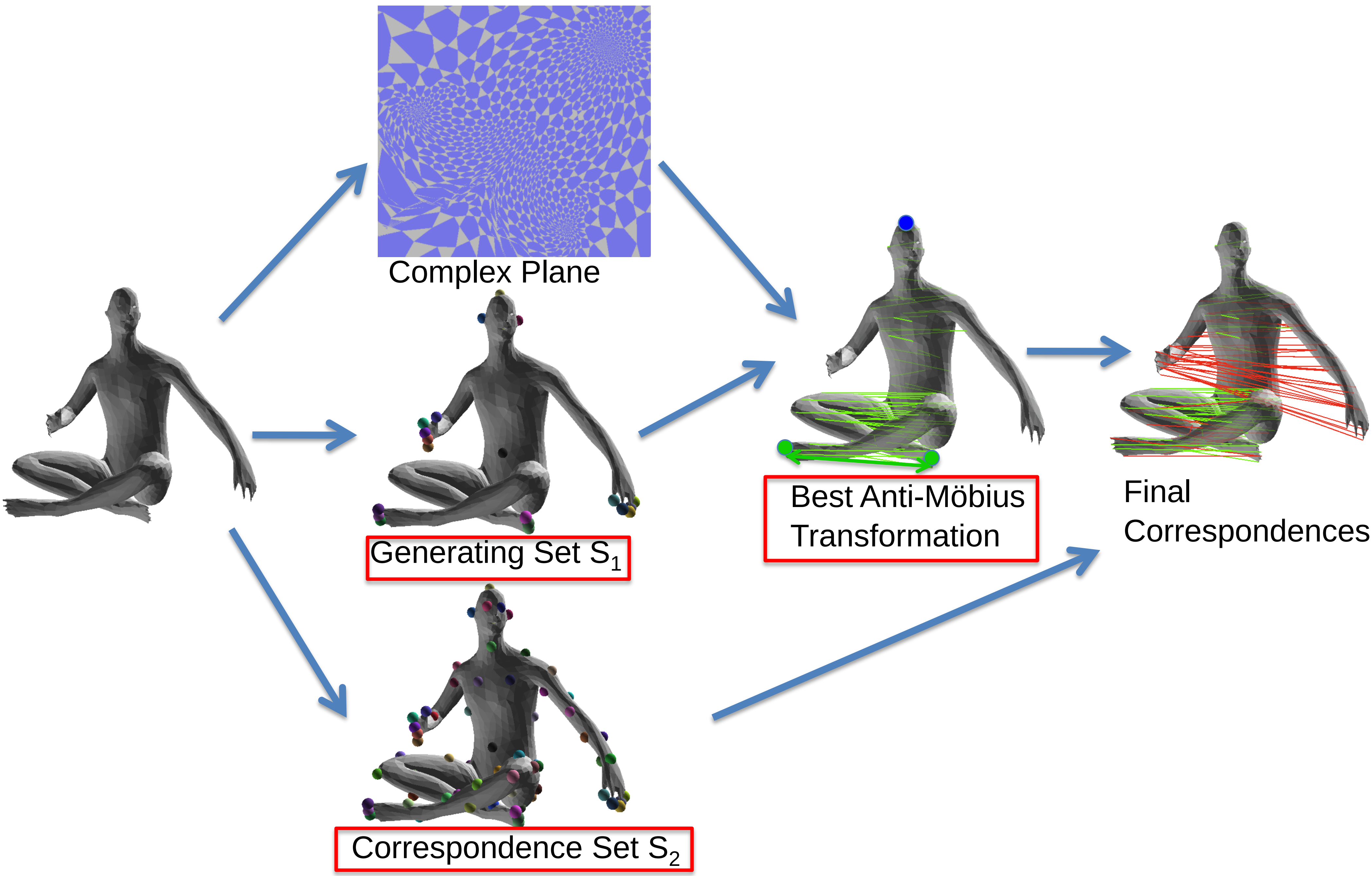
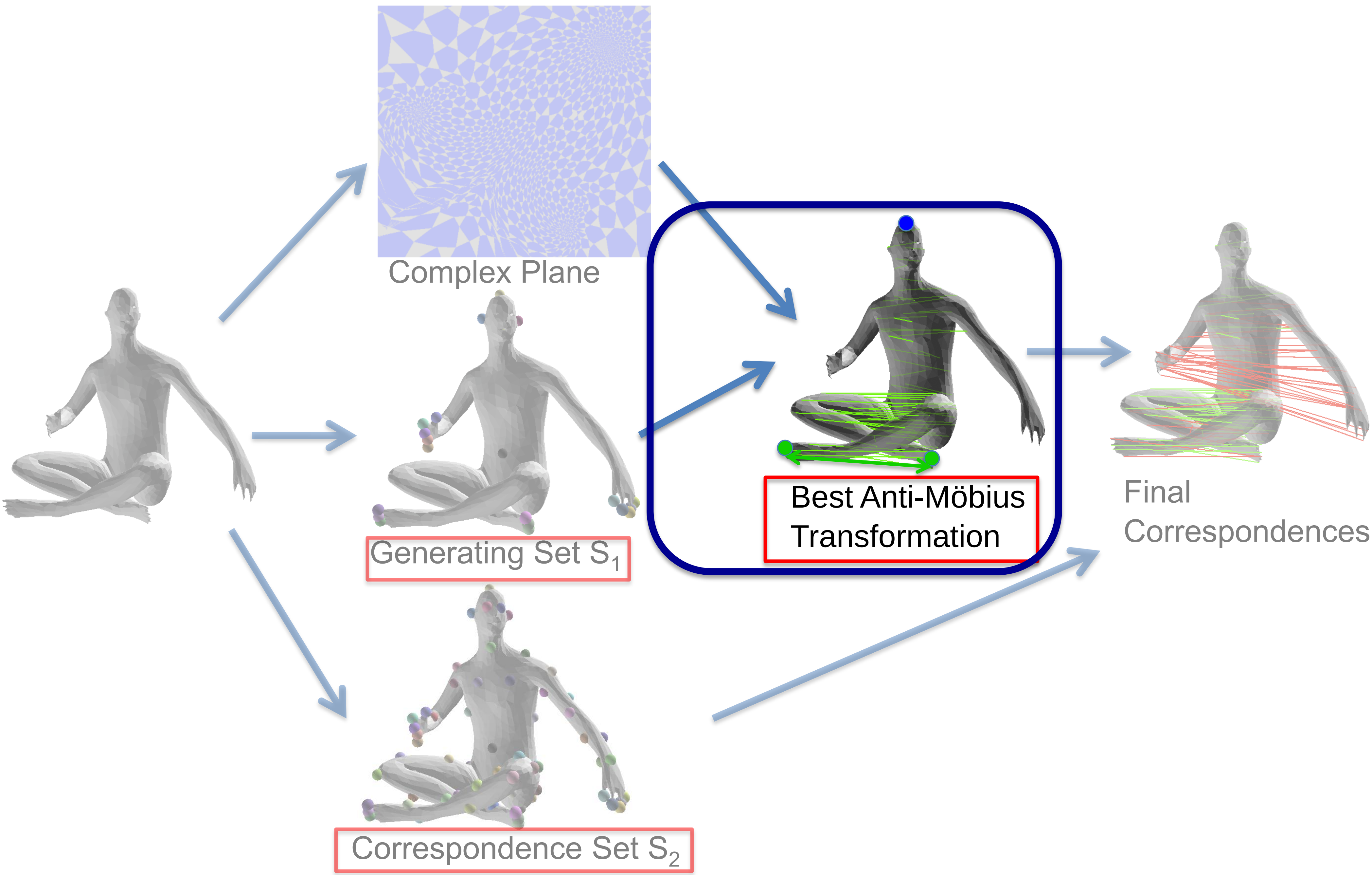
- Theorem 7.6:
  - If f is bilateral reflective, the gradient of Φ is parallel to the curve of stationary points of f

    At least 2 stationary points will have $\nabla|_p \Phi = 0$
  - For any other symmetry if there is a stationary point it would be a critical point of Φ

Pipeline


Complex Plane
Generating Set S_1
Correspondence Set S_2
Best Anti-Möbius Transformation
Final Correspondences

Pipeline



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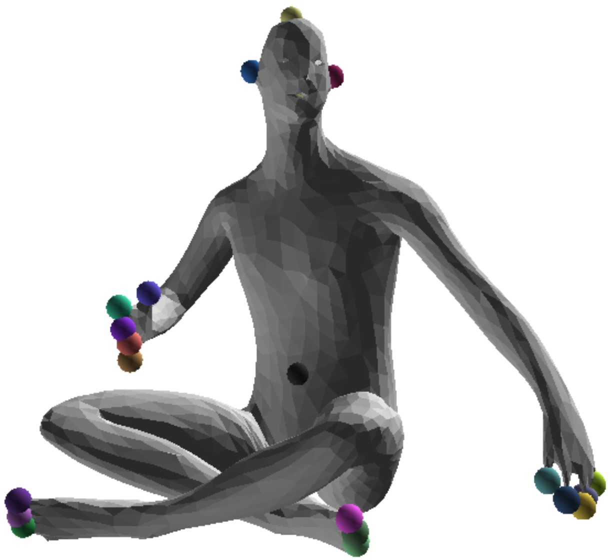
Searching for the Best Anti-Möbius Transformation

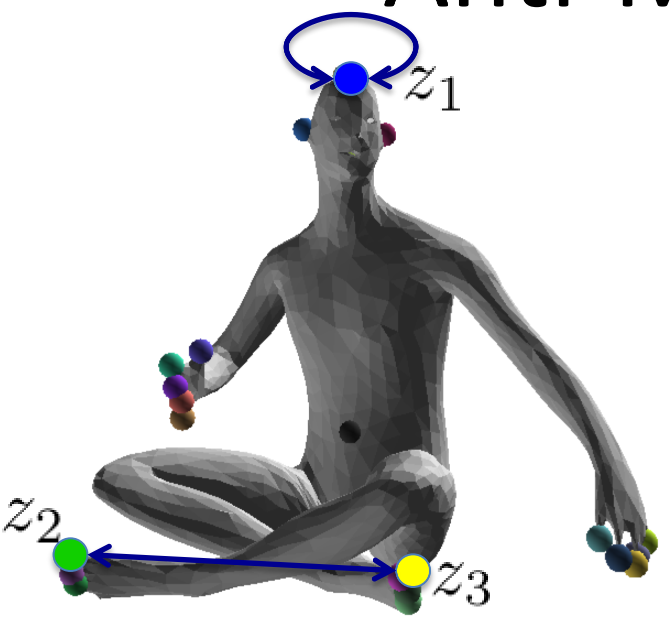


Symmetry Invariant Point Set from AGD (21 points)

- Goal: find a conformal map that is as isometric as possible

- Approach: use small symmetry invariant set to explore conformal mappings

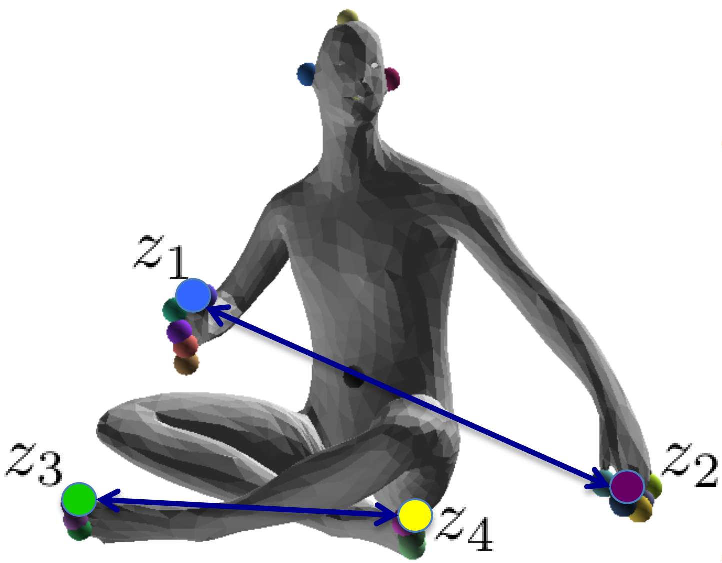
Searching for the Best Anti-Möbius Transformation



Symmetry Invariant Point Set from AGD (21 points)

- **Explore all 3-plets:** $\boxed{\begin{array}{l} z_1 \longrightarrow z_1 \\ z_2 \longrightarrow z_3 \\ z_3 \longrightarrow z_2 \end{array}}$
- Explore all 4-plets: $\begin{array}{l} z_1 \longrightarrow z_2 \\ z_2 \longrightarrow z_1 \\ z_3 \longrightarrow z_4 \\ z_4 \longrightarrow z_3 \end{array}$

Searching for the Best Anti-Möbius Transformation



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- Explore all 3-plets:

$$z_1 \longrightarrow z_1$$
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- **Explore all 4-plets:**

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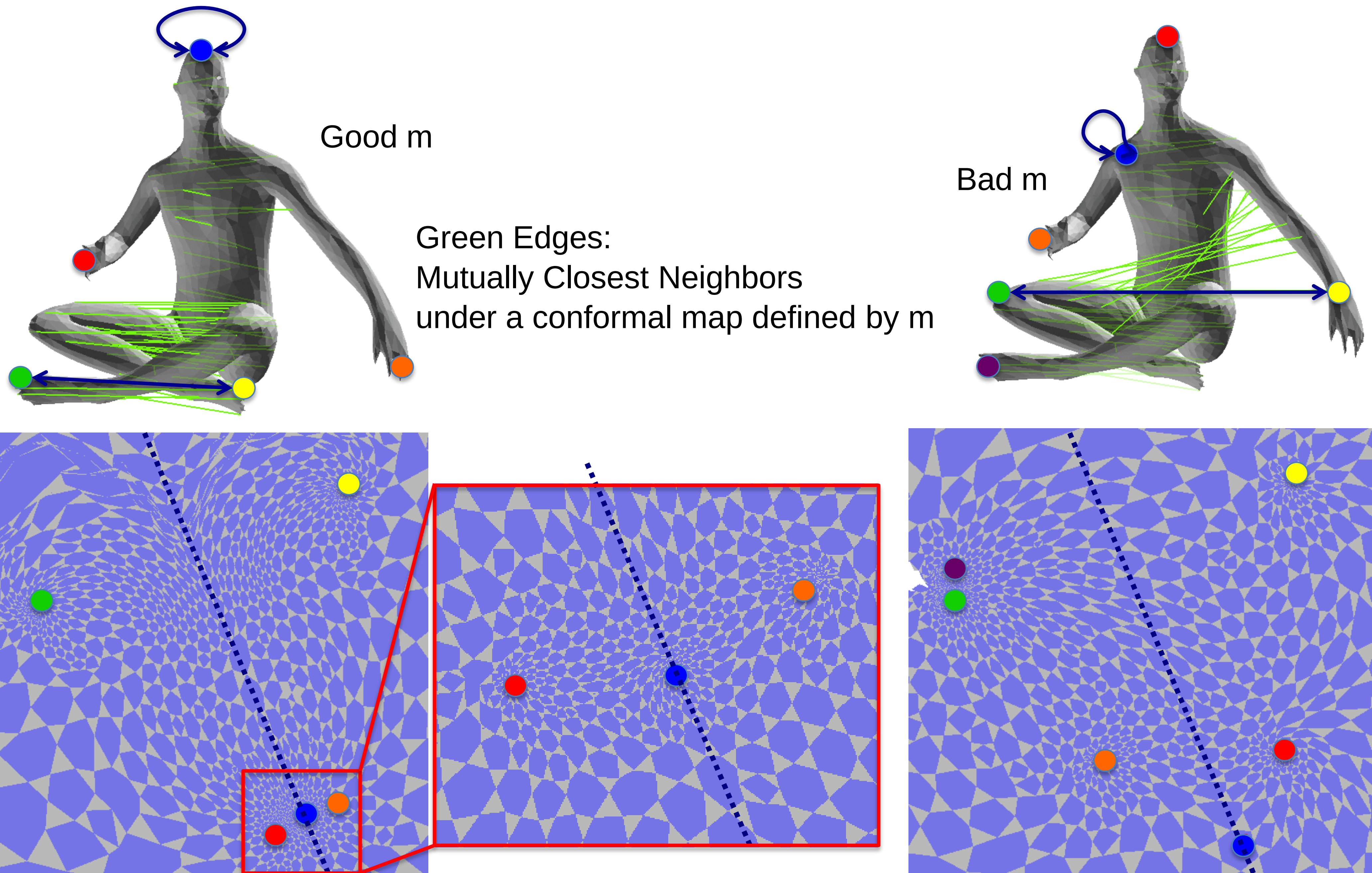
Best Anti-Mobius Transformation



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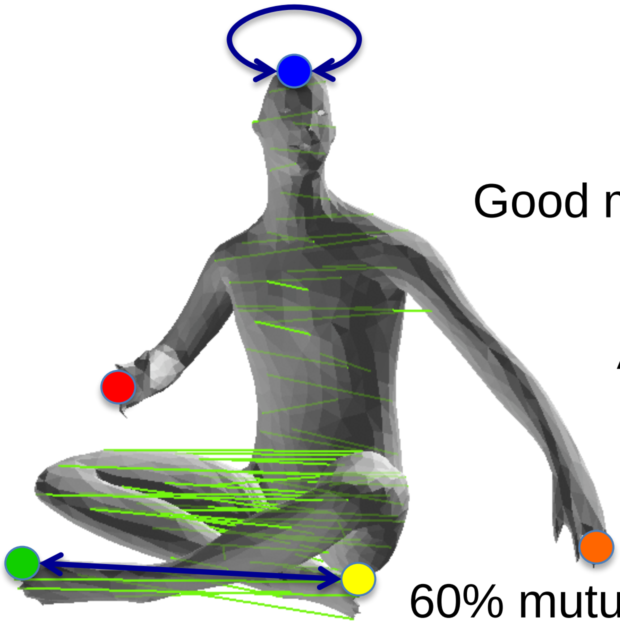
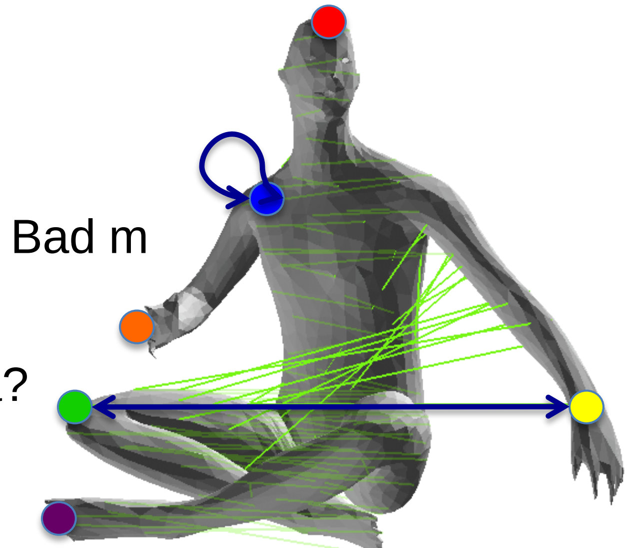


Good m

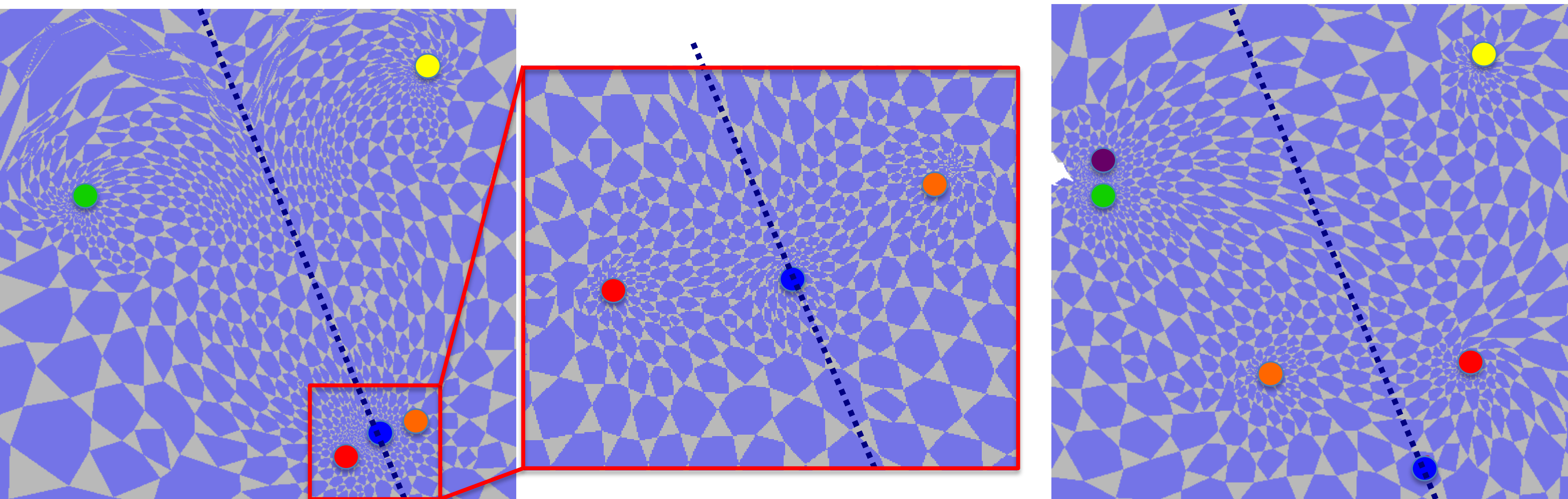
Alignment Score:
How well does the map preserve area?



Bad m

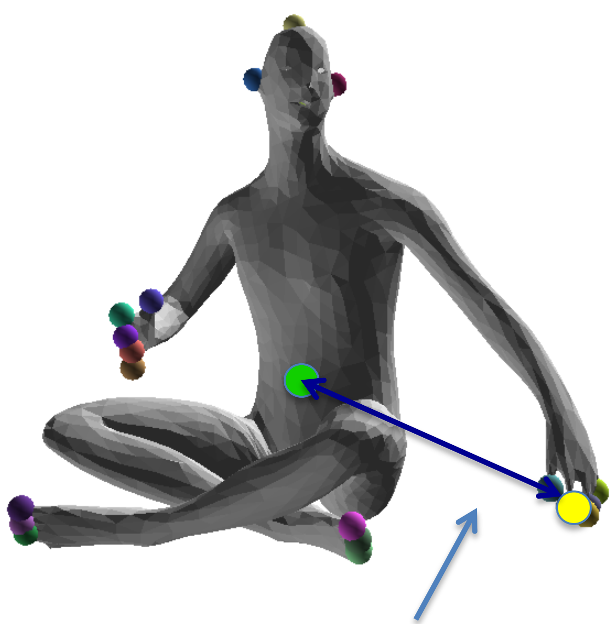
60% mutually closest

17% mutually closest



Pruning

- **Ignore a-priory bad mappings**
  - **Different AGD values**
  - Too close correspondences
  - Different geodesic distances



Bad correspondence

Pruning

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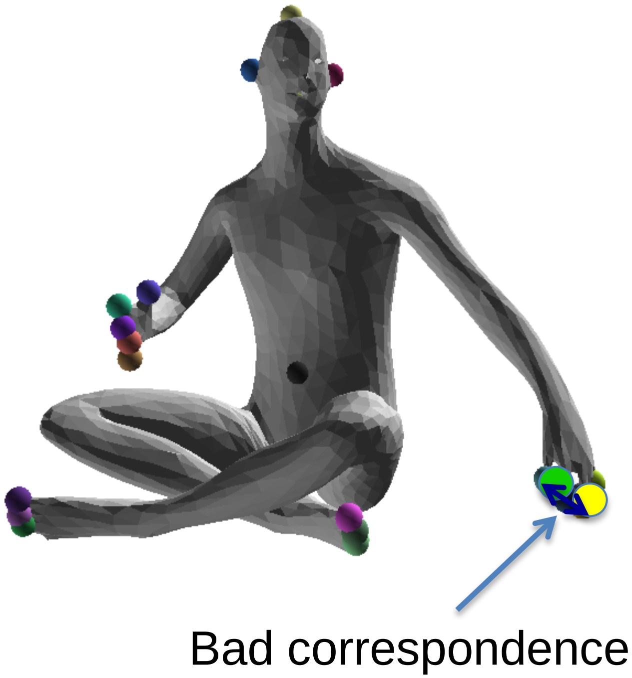

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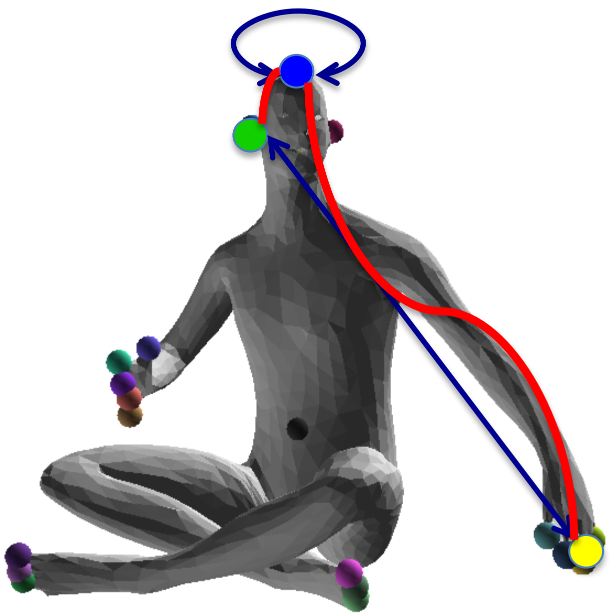
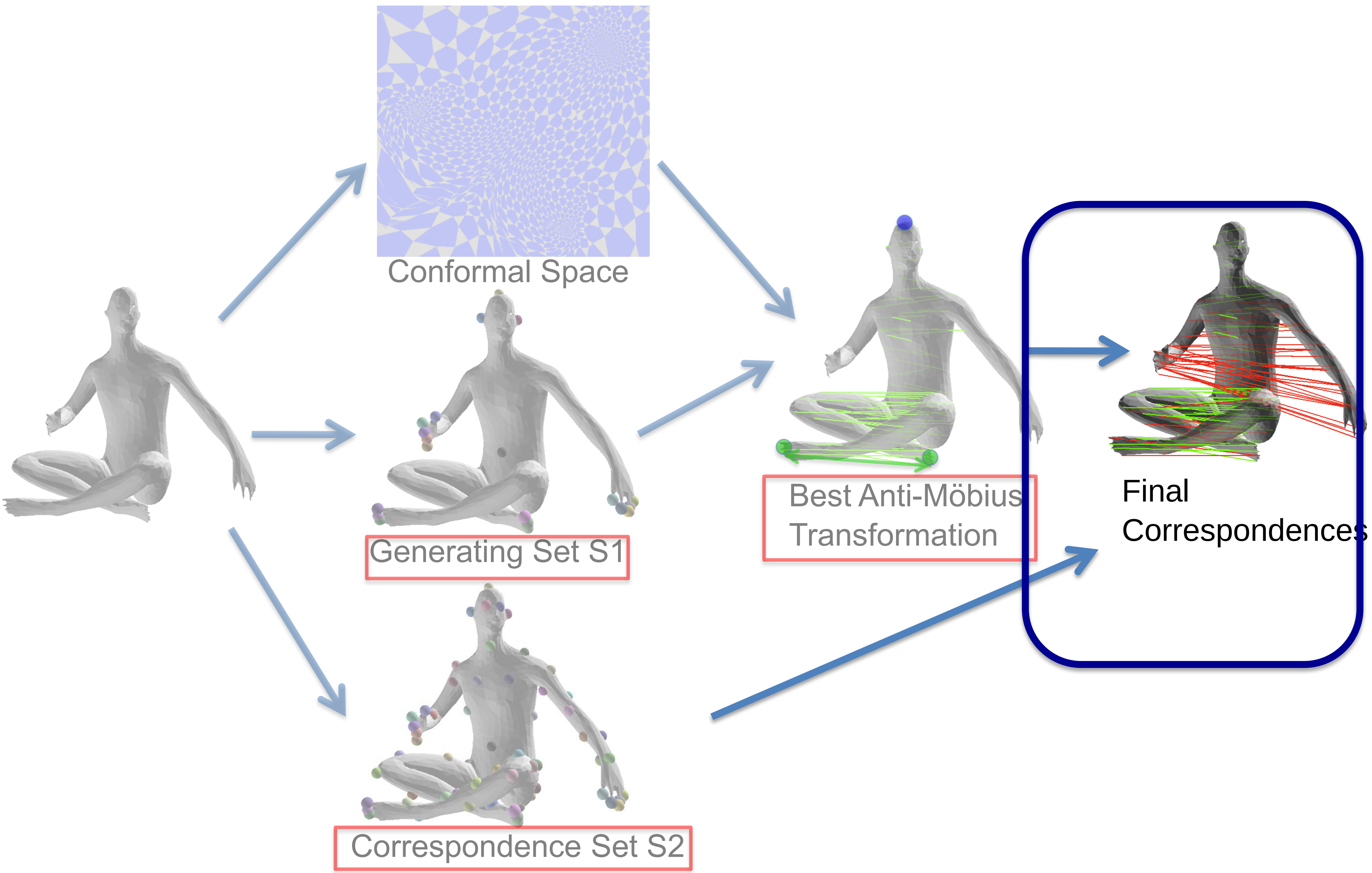


Bad Triplet

Pipeline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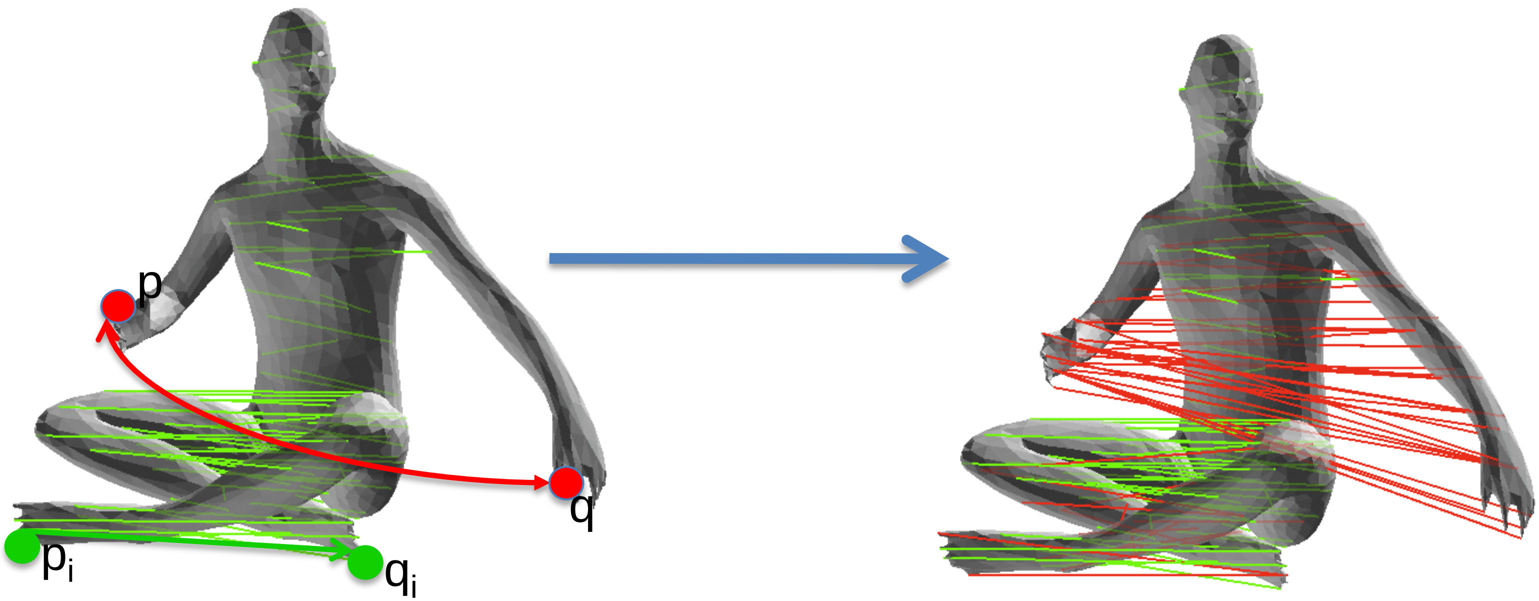
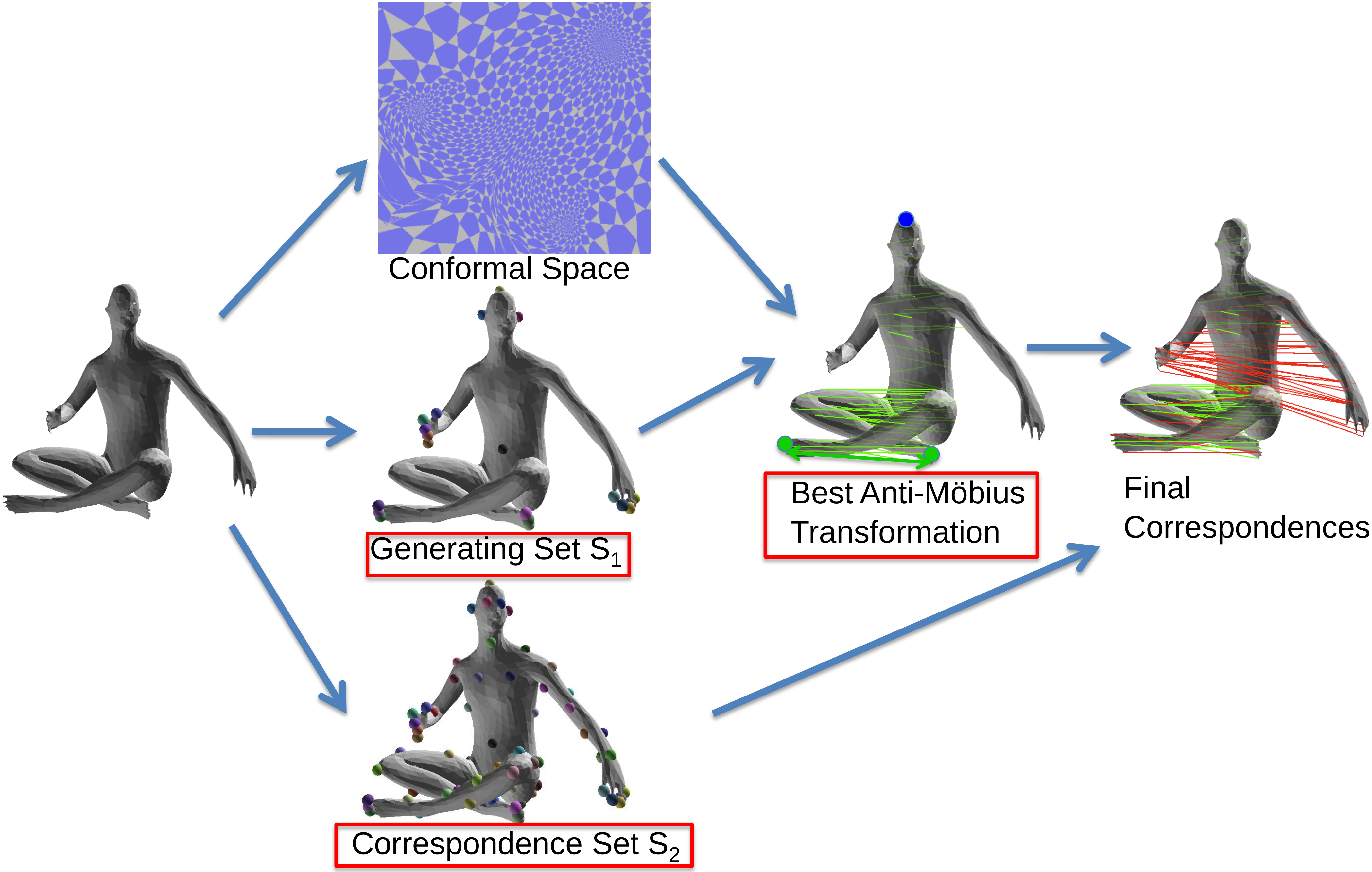


- Goal: Given sparse correspondences: $(p_i, m(p_i) = q_i)$ find a correspondence q for every p
- Approach: For any p, find q so that their geodesic distances to sparse set are same

Similar to:
"Efficient computation of isometry-invariant distances between surfaces". Bronstein et al. 2006

Pipeline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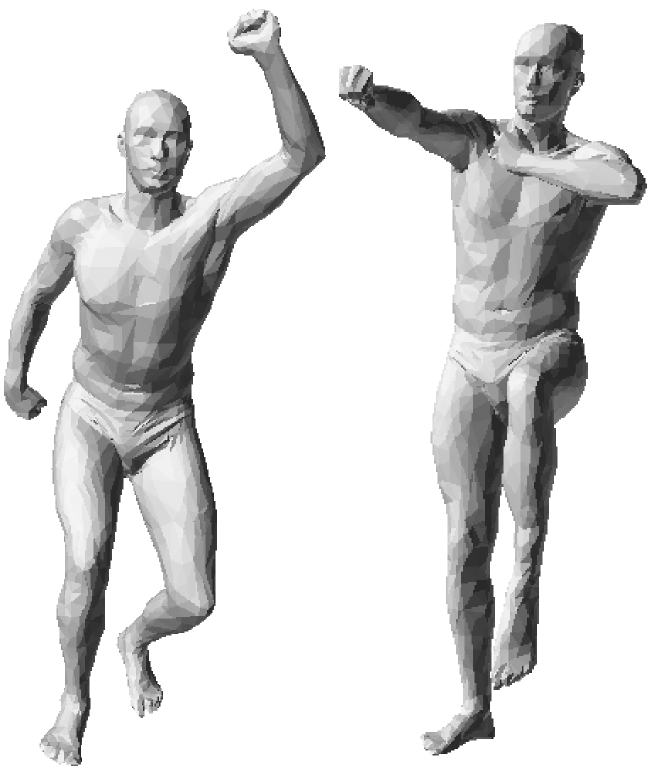
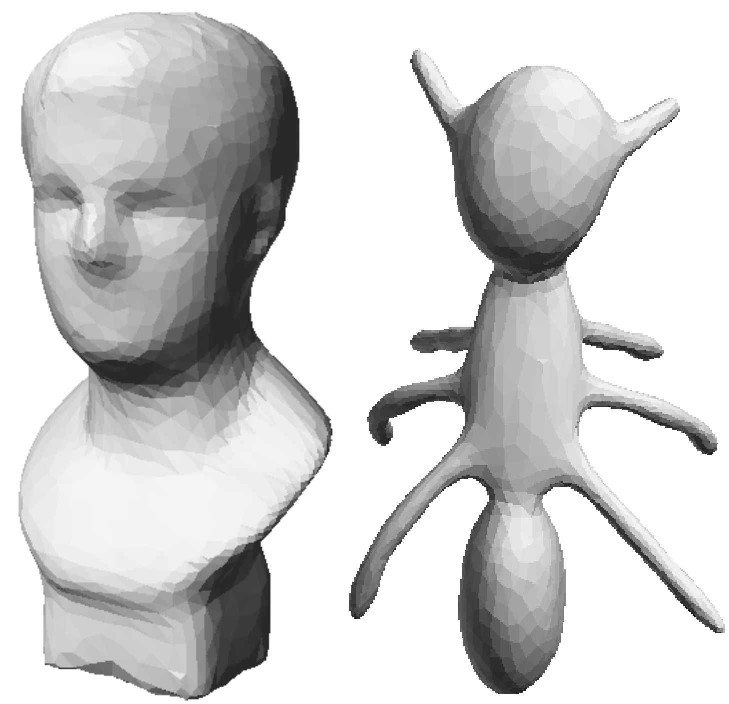
Results
Benchmark

- Goal: quantitatively evaluate performance of our method on 366 models



Scape:
71 Models

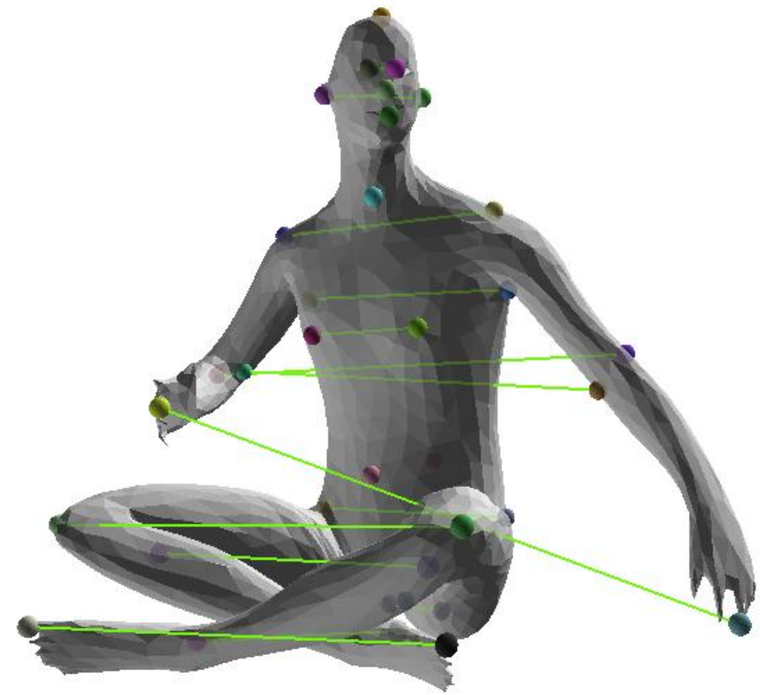
Non-Rigid World:
75 Models



SHREC, Watertight'07:
220 models

Results
Benchmark

- **Ground Truth**
- Geodesic Error
- Correspondence Rate
- Mesh Rate
- Results



$f_{\mathrm{true}} : S_{\mathrm{true}} \to S_{\mathrm{true}}$

Results
Benchmark

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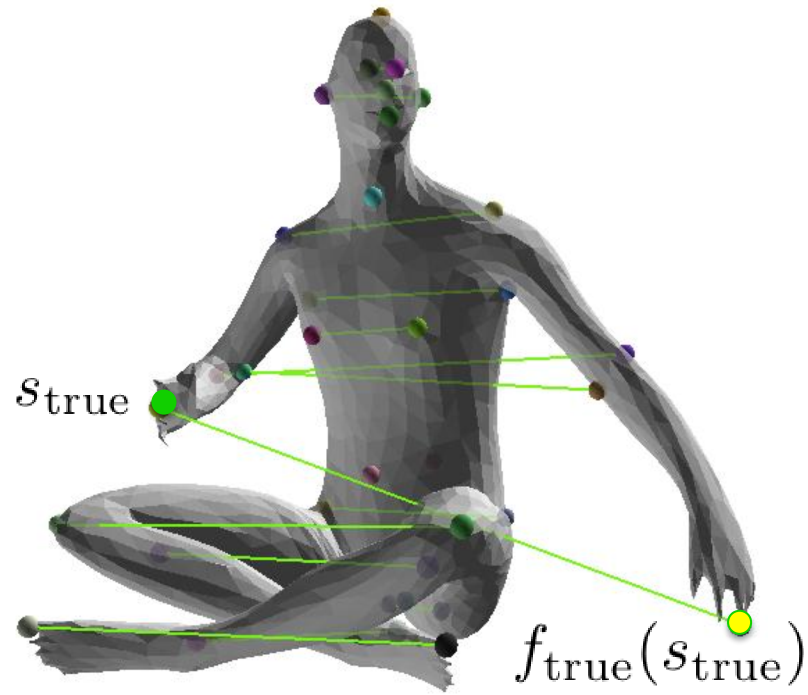

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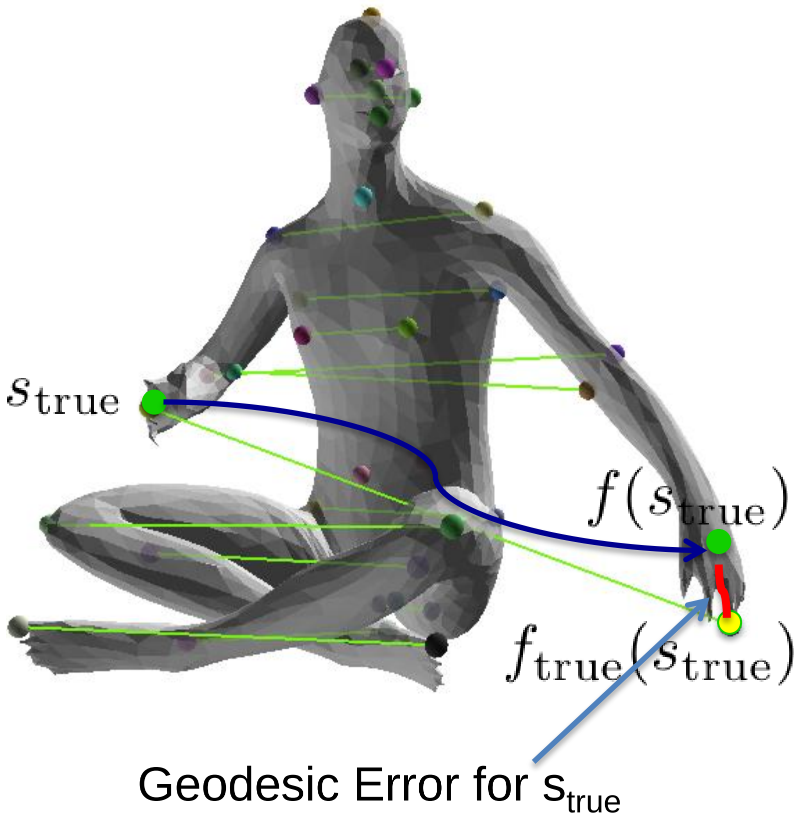

$$\sum_{s_{\mathrm{true}} \in S_{\mathrm{true}}} d_g(f(s_{\mathrm{true}}), f_{\mathrm{true}}(s_{\mathrm{true}}))$$

Results
Benchmark

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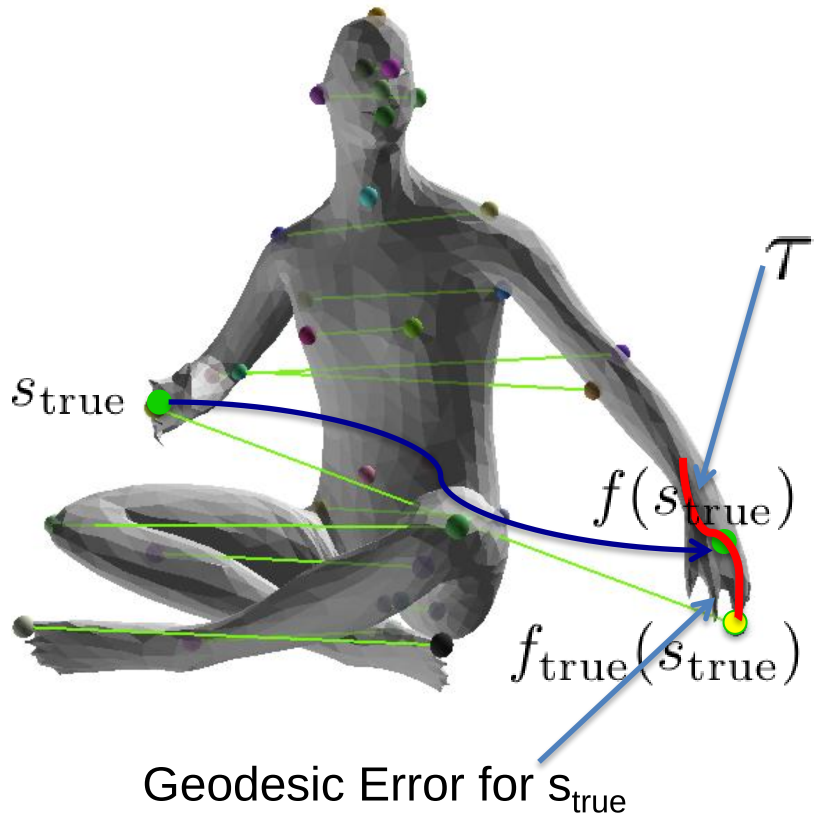

$$d_g(f(s_{\text{true}}), f_{\text{true}}(s_{\text{true}})) < \tau$$

Results
Benchmark

- Ground Truth
- Geodesic Error
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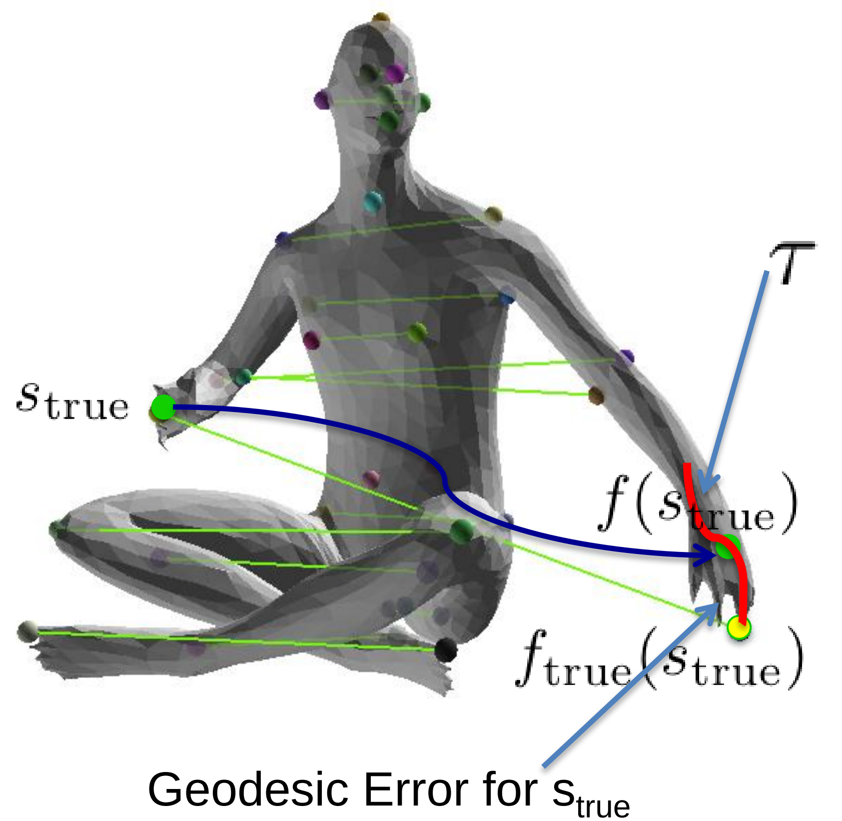
- **Mesh Rate**
- Results

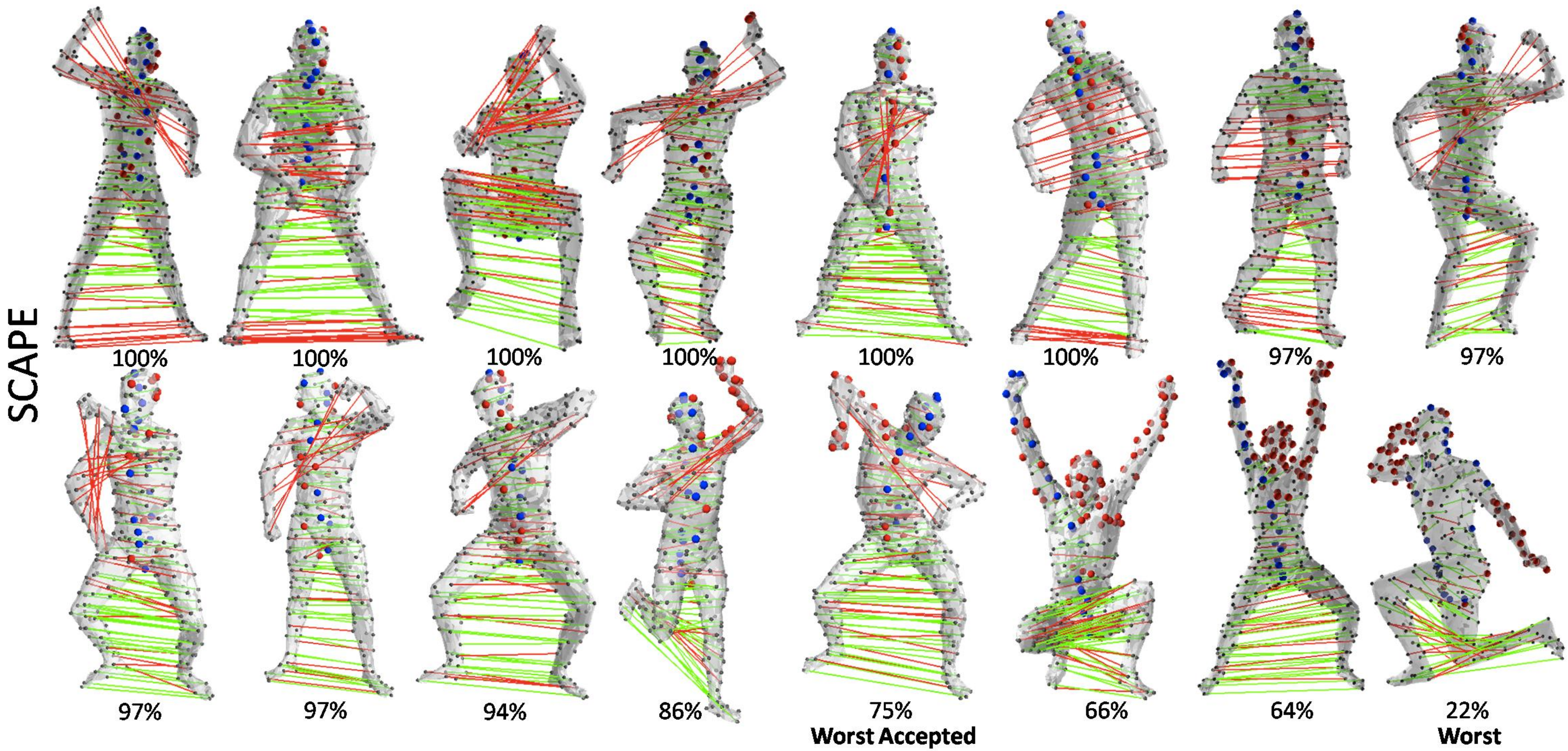


Correspondence Rate > 75%

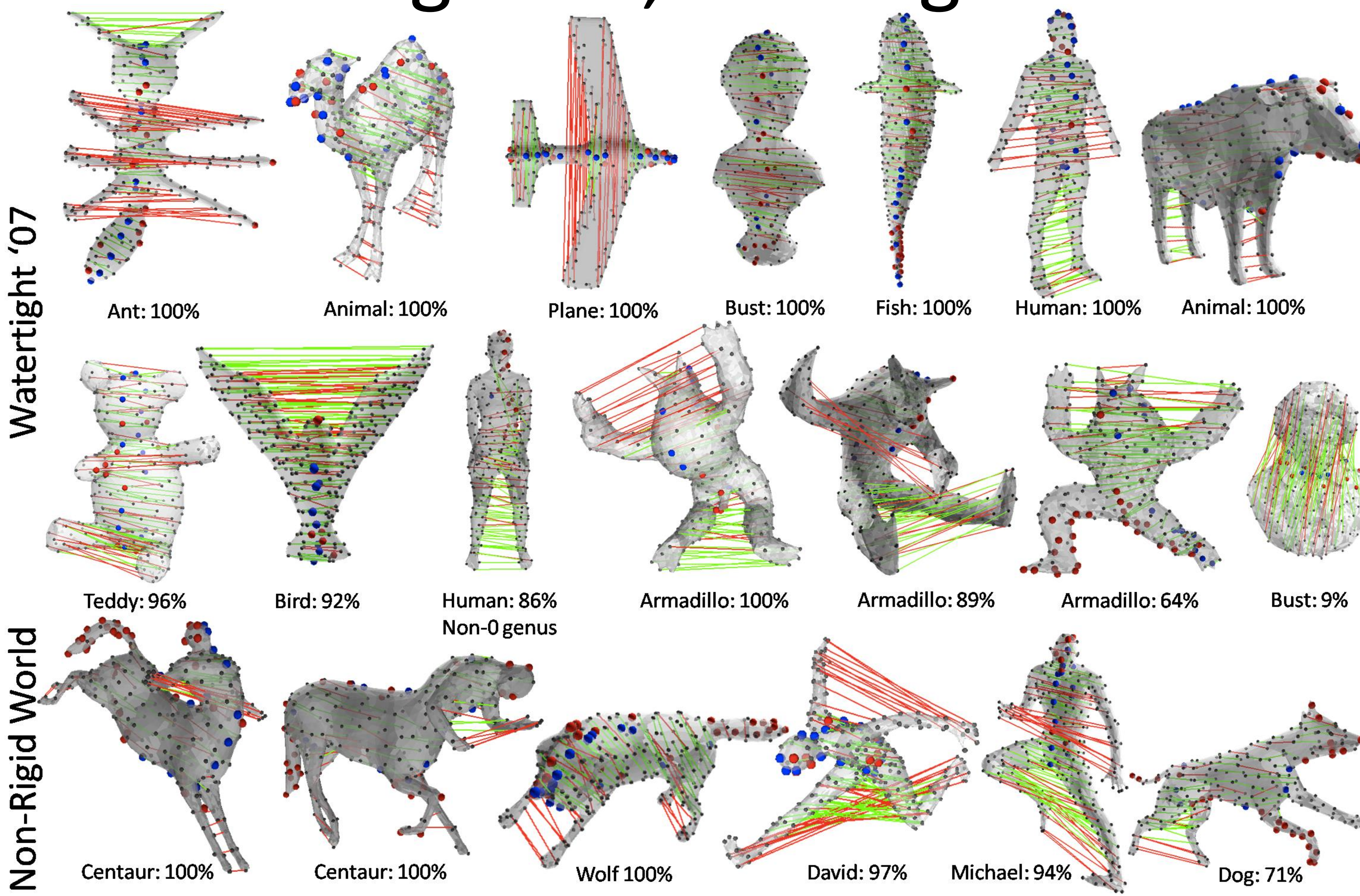
Results
Benchmark

	Non-Rigid World	SCAPE Human	SHREC Watertight	All Data Sets
Geodesic	3.3	4.2	1.93	2.65
Corr rate	85%	82%	83%	83%
Mesh rate	76%	72%	75%	75%

Results
Scape



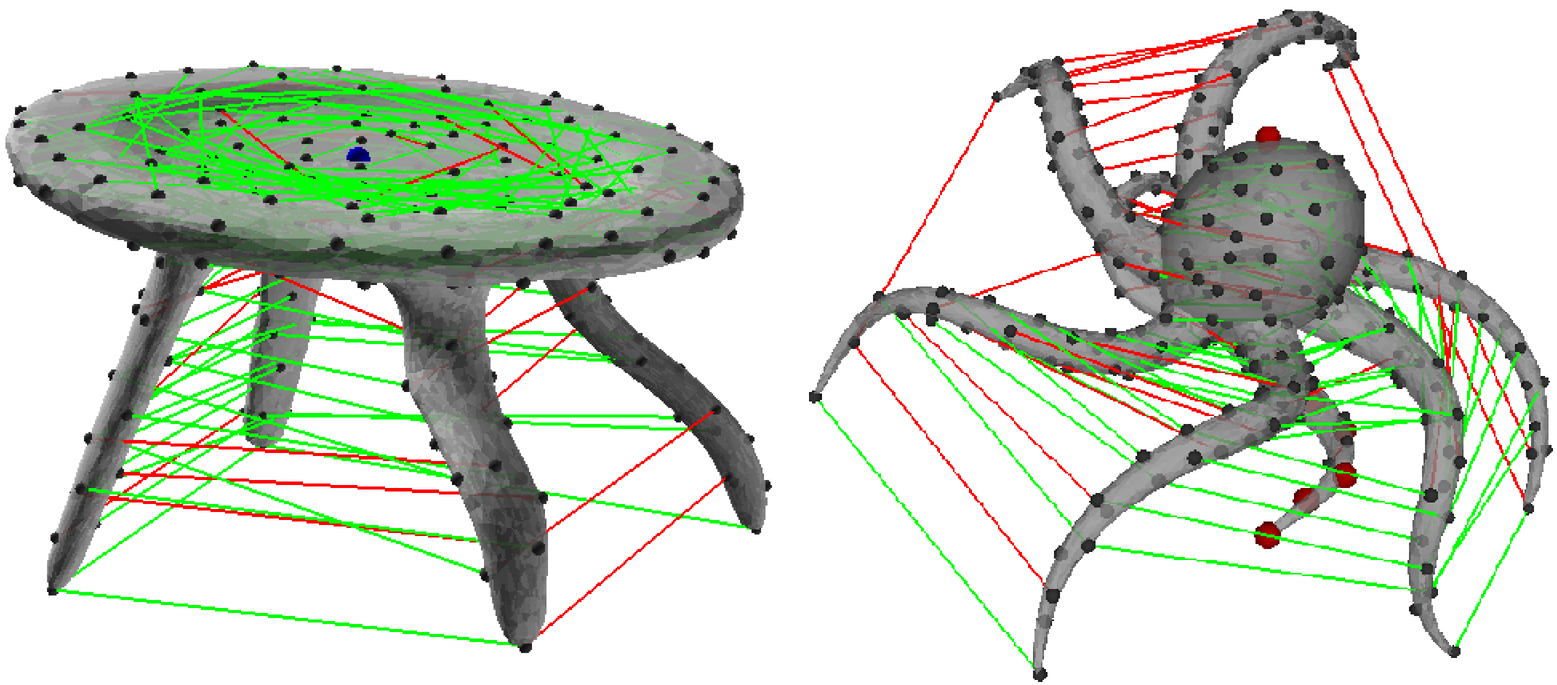
Results
Watertight'07, Non-rigid world



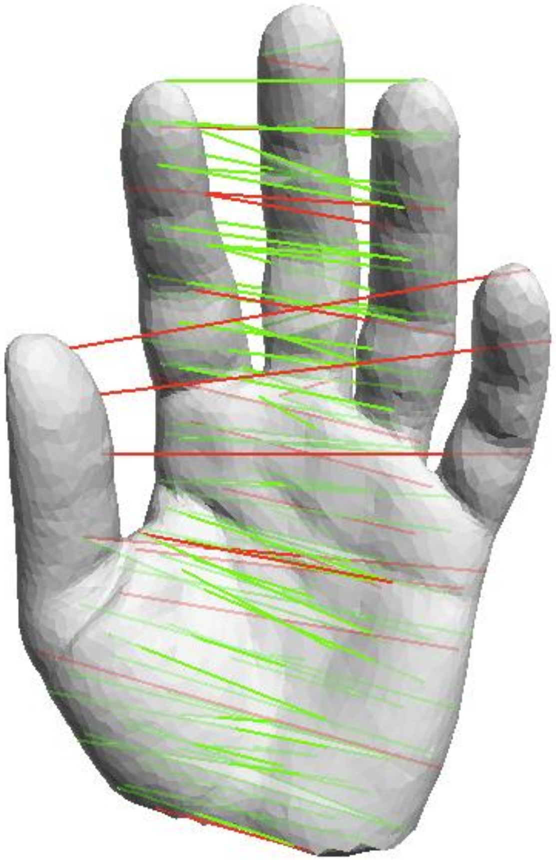
Comparison

	Our Proposed Method	Mobius Voting (Lipman '09)
Geodesic	3.49	6.78
Corr rate (%)	86%	70%
Mesh rate (%)	72%	51%
Time (s)	25s	310s

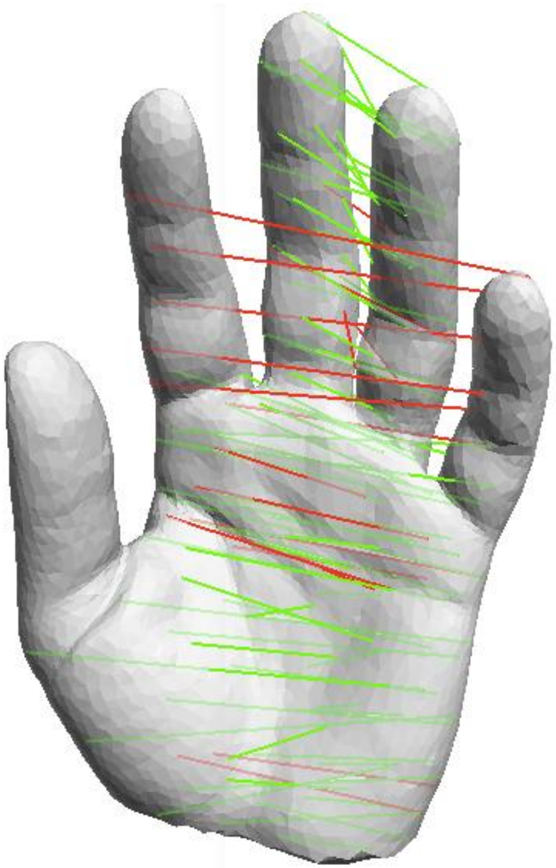
Rotational Symmetry



Large-scale outliers



Best Mobius



Second Best Mobius

Conclusion

- Anti-Mobius Transformations can be used for analysis of intrinsic symmetries
- Method succeeded on 75% of 366 meshes
- Our method improves speed and performance significantly over Möbius Voting

Limitations

- General partial intrinsic symmetries
  - Alignment error for a conformal map is global
- Symmetry-invariant sets
  - Robustness to noise
  - Various functions (other than AGD)

Acknowledgements

- Funding
  - NSF (IIS-0612231, CNS-0831374, CCF-0702672, and CCF-0937139)
  - NSERC Graduate Scholarship (PGS-M, PGS-D)
  - Google
  - Rothschild Foundation
- Data
  - Daniela Giorgi and AIM@SHAPE (Watertight'07)
  - Drago Arguelov and Stanford University (SCAPE)
  - Project TOSCA (Non-Rigid World)

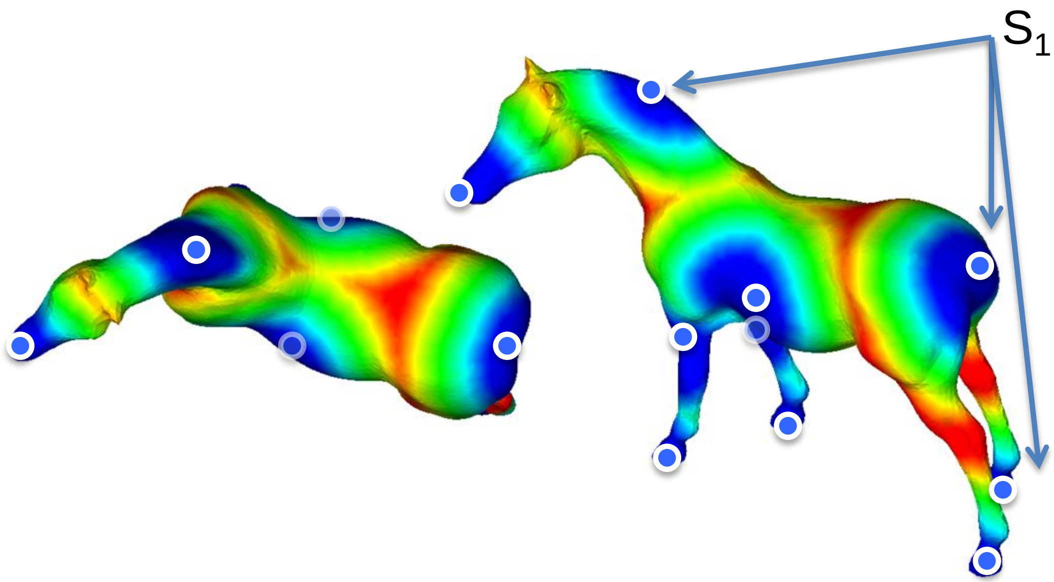
Online

- More data and results:

  http://www.cs.princeton.edu/~vk/IntrinsicSymmetry/

Finding a Symmetric Point Set

- Minimal Geodesic Distance $\Phi_{\mathrm{mgd}}(p; S_1) = \min_{q \in S_1} d_g(p, q)$



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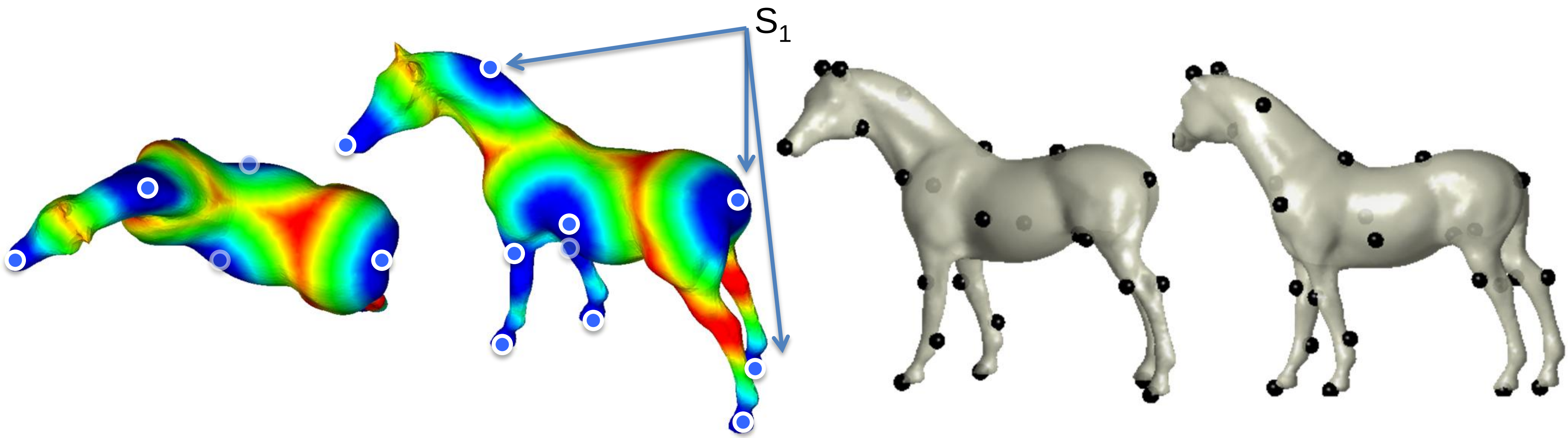

  - Can apply iteratively to construct set of arbitrary size
  - Less robust
  - Correspondence Set